J. Fluid Mech. (1972), *vol.* 52, *part* 2, *pp.* 209–243
Printed in Great Britain

On Long's hypothesis of no upstream influence in uniformly stratified or rotating flow

By MICHAEL E. McINTYRE

Department of Applied Mathematics and Theoretical Physics,
University of Cambridge

(Received 11 December 1970 and in revised form 23 December 1971)

The weakly nonlinear, two-dimensional problem for the disturbance due to a slender obstacle in a uniformly stratified, Boussinesq fluid moving past the obstacle with constant basic horizontal velocity U, is considered up to second order in the amplitude ϵ of the disturbance. Analogous rotating problems are also treated. Particular attention is given to calculating explicitly the columnar-disturbance strengths upstream and downstream of the obstacle, both in the stratified and in the rotating problems, with a view to discussing the truth or otherwise of Long's hypothesis (LH).

Whether or not columnar disturbances are found far upstream, violating LH, depends, *inter alia*, on whether or not the flow is externally bounded by rigid horizontal planes (or by a tube or annulus, in the rotating problem), and on whether the problem is made determinate by means of an 'inviscid transient' formulation, or by means of a 'viscous' one.

The inviscid, transient, bounded problem, for time-development of lee waves from a state of no initial disturbance, always exhibits columnar disturbances of order ϵ^2 somewhere in the fluid. They are generated, not near the obstacle, but in the 'tails' or transient terminal zones of the lee-wave trains. The columnar-disturbance strengths are largely independent of how the flow is set up from an initially undisturbed state. In all but one instance the effect is non-zero far upstream. The exception is the singly-subcritical stratified (or narrow-gap rotating) case, in which the excitation has modal structure $\sin(2z)$, the fluid region being $0 \leqslant z \leqslant \pi$; in this case the only columnar disturbance that can penetrate upstream has structure $\sin z$ and so is not excited.

A completely different result holds for 'viscous' formulations for unseparated, bounded régimes (with steady lee waves spatially attenuated by effects of small molecular diffusion). The strengths of all columnar disturbances, upstream and downstream, vanish in the limit of small diffusivity.

In the inviscid, transient, unbounded problem, the upstream influence is, likewise, evanescent, being $O(\epsilon^2 t^{-2})$ as time $t \to \infty$.

The basic expansion in powers of ϵ will be invalid for times $\propto \epsilon^{-1}$ or greater, because of resonant-interactive instability of the lee waves.

CONTENTS

1. Introduction

This paper is concerned with the weakly nonlinear lee-wave régime for two-dimensional disturbances in a Boussinesq, incompressible, stably stratified fluid characterized by constant basic buoyancy frequency

$$N = \{-g\,\partial \ln(\text{basic density})/\partial z\}^{\frac{1}{2}},$$

and also for analogous homogeneous, rotating fluid systems. The disturbances are caused by a stationary slender obstacle past which there is uniform basic flow

$$(U, 0, 0).$$

The co-ordinates (x, y, z) are oriented so that for stratified systems the acceleration due to gravity is $(0, 0, -g)$, or so that for homogeneous systems the basic rotation is $(\frac{1}{2}N, 0, 0)$.

Except in §6 it is supposed that the fluid, as well as being inviscid and non-diffusive, is confined between boundaries z = constant. Then there are discrete disturbance modes

$$f_n(z)\exp[ik\{x-(U+c_n)t\}] \tag{1.1}$$

each of which has intrinsic phase and group velocities

$$c_n(k), \quad \gamma_n(k) \equiv \partial_k(kc_n)$$

that are equal at $k = 0$ and of decreasing magnitude (so that $\gamma_n/c_n < 1$) for $k > 0$. At given n this dispersion property allows stationary lee waves to appear downstream of the forcing effect when, and only when, it also allows x-independent 'columnar' steady disturbances to penetrate upstream. Excitation of those columnar disturbances that can penetrate upstream implies permanently altered velocity and density profiles arbitrarily far upstream (Trustrum 1964; Greenspan 1968).

This possibility of 'upstream influence' is known to be realized, for instance, in two-dimensional, stratified flow over a shallow step or semi-infinite plateau, according to linearized theory for infinitesimal values of an appropriate dimensionless measure of the step height,

$$\epsilon \ll 1.$$

Other examples are axisymmetric rotating flow past a semi-infinite cavity, or any other source-like forcing effect (Benjamin & Barnard 1964; Trustrum 1964, 1971; Wong & Kao 1970). A lee-wave régime for which theory predicts excitation of columnar disturbances at first order in ϵ will be said to be *of type one*.†

In the dipole-like case, for example a slender obstacle of finite extent in x, linearized theory predicts zero excitation of columnar disturbances by the obstacle so that, in this case, there is no upstream influence at first order in ϵ. We then speak of lee-wave régimes *of type zero*. Long's hypothesis (LH) is the stronger statement that the upstream influence is *exactly* zero, or zero to all orders in ϵ, in problems of type zero. LH has been a subject of recent controversy, both for the bounded problem and also for the problem in which the external boundaries are removed to $z = \infty$ (Stewartson 1970, and references). The truth of LH is a necessary condition for the physical relevance of certain exact, finite-amplitude solutions representing stationary lee-wave patterns (Yih 1965, Greenspan 1968, Benjamin 1970); these 'Long-type' exact solutions apply, in the case of a Boussinesq fluid, only if the flow at upstream infinity is uniform. It should be noted that the foregoing tacitly assumes a suitable context which renders the problem determinate, such as that of inviscid evolution from an undisturbed, uniform initial state, or that of steady, viscous, diffusive flow in the limit of small diffusivities.

The recent work of Benjamin (1970) strongly suggests invalidity of LH for the bounded problem, in the inviscid, transient context. His paper demonstrates indirectly that columnar disturbances of order ϵ^2 must occur somewhere in the

† 'Upstream influence' is also a feature of the blocked or Taylor–Proudman régime (U infinitesimal, ϵ finite: Morgan 1951; Stewartson 1952; Bretherton 1967), but connection with the present lee-wave régimes appears remote and would seem likely to be superficial as well. Intervening régimes seem inaccessible to rational theory, and appear turbulent experimentally. The lack of connection will be further emphasized by the entirely different manner in which columnar disturbances turn out to be generated in the cases to be considered below, and the fact that the alteration to the velocity profile can be opposite in sense to that which would be expected from 'blocking' (§4.5 below).

flow field, by showing that the wave drag on the obstacle cannot otherwise be accounted for in terms of the rate of increase of fluid impulse. Despite a claim made in the paper, the impulse argument does not quite settle the LH issue, because of the logical possibility that all the second-order columnar disturbances might appear far downstream.

The purpose of the present work is to clarify this situation by explicitly solving for the relevant effects of order ϵ^2. Most attention is given to the inviscid, bounded problem considered by Benjamin; but the corresponding qualitative results in two other possible contexts for LH become clear also, and are stated in §6. For the inviscid, transient, bounded problem we confirm Benjamin's result that columnar disturbances are always excited when lee waves are present. Further, we give the rules for calculating their strengths and locations, both in the stratified (§§4.5, 4.6) and in the rotating-tube problem (appendix A)†. It is argued (§4.6) that the columnar-disturbance strengths are largely independent of how the lee-wave régime is established, as long as it is established from an initially undisturbed state. Upstream influence usually occurs; but in one instance it is absent and the second-order columnar disturbances *are* confined downstream of the obstacle, together with the corresponding fluid impulse. (This can also occur for lee-wave régimes in a system consisting of two layers of homogeneous fluid, as has been shown recently by Keady (1971).)

An interesting feature, and incidentally one that underlines the lack of connection with the blocked-flow régime, is that the excitation of columnar disturbances takes place not in the near field, but in the far field of the obstacle. For the Long-type basic flow under consideration, the columnar disturbances emanate *from the 'tails', or transient terminal zones, of the lee-wave trains*. The importance of the lee-wave 'tails' is evident as soon as one realizes that they represent the only reasonable possibility. The existence of the Long-type steady solution, which is of linearized form but which satisfies the exact equations, tells us in advance that, when we make an expansion in powers of ϵ, no steady part of the order-ϵ lee-wave system can be responsible for the order-ϵ^2 effect that Benjamin's impulse argument assures us must occur in association with the lee waves. (In Benjamin's averaged equation (4.25), the effect of the lee-wave tails would appear on the right-hand side as an inhomogeneous forcing function involving delta functions and their derivatives. This would be the counterpart, viewed on a different length scale, of the right-hand sides of equations (4.1 *a*, *b*) below.)

From a general standpoint, the role of the lee-wave tails can be regarded as another illustration of the importance of slow spatial and temporal variations in the amplitude of a wavetrain, when considering phenomena of second order in amplitude. This has been appreciated for some time in the case of surface gravity waves (Longuet-Higgins & Stewart 1964). Another, more closely related instance of it is to be found in the recent discussion by Bretherton (1969) of the second-order motions induced by freely propagating internal-gravity-wave packets, in an unbounded medium. One of the problems solved by Bretherton anticipates, in important respects, the discussion given in §6.2 for the unbounded problem.

† The latter results are developed in greater detail in a forthcoming paper by Miles (1972), with particular attention to the asymptotic approach to the unbounded problem.

Throughout the body of this paper we shall use descriptive language appropriate to a Boussinesq, thermally stratified fluid; but the rotating analogy should be borne in mind. The reader already familiar with the linearized, transient lee-wave problem may prefer to omit §3 and proceed to §4. Our main results are stated in §§4.5, 4.6, 6, and appendix A. An important restriction on their range of validity is indicated in §7.

2. Formulation

2.1. *Equations and boundary conditions*

The equations will be made dimensionless with the use of U as velocity scale, D as length scale, and D/U as time scale. Then the buoyancy frequency will appear as

$$K \equiv ND/U.$$

We assume two-dimensional motion of an inviscid, Boussinesq, incompressible fluid, except in §6 where viscous effects will be discussed. The word 'inviscid' signifies that the viscosity and the diffusivity of density anomalies are both assumed zero. The buoyancy acceleration is $(0, 0, -g\rho'/\rho_0)$, where ρ' is the (small) departure from the basic-state density

$$\rho_b = \rho_0(1-g^{-1}N^2z) \quad (g^{-1}N^2D \ll 1).$$

The dimensionless form of the buoyancy acceleration is $(0, 0, \theta)$, where

$$\theta = -\frac{D}{U^2}\frac{g\rho'}{\rho_0}.$$

A disturbance stream function is defined by

$$\left.\begin{aligned} x \text{ component of velocity} &= 1+\partial_z\psi, \\ z \text{ component of velocity} &= -\partial_x\psi, \end{aligned}\right\} \tag{2.1}$$

so that the y component of vorticity is

$$\eta \equiv \nabla^2\psi \equiv (\partial_x^2+\partial_z^2)\,\psi. \tag{2.2}$$

The equations can then be written without further approximation as

$$(\partial_t+\partial_x)\,\eta+\partial_x\theta = \partial(\psi,\eta)/\partial(x,z), \tag{2.3a}$$

$$(\partial_t+\partial_x)\,\theta-K^2\partial_x\psi = \partial(\psi,\theta)/\partial(x,z). \tag{2.3b}$$

Let

$$\psi = \theta = 0 \quad \text{for} \quad t<0, \tag{2.4}$$

and let the introduction of the obstacle over which the fluid is to flow be described by

$$z = \epsilon \mathsf{h}(x,t) \quad (\epsilon \ll 1) \tag{2.5}$$

where

$$\mathsf{h} = 0 \quad \text{for} \quad t<0.$$

It is convenient to suppose that, *qua* function of x, h has certain weak smoothness properties to be made precise by (3.5*b*) below, and that

$$\mathsf{h} = O(e^{-\beta|x|}) \quad \text{as} \quad |x|\to\infty, \tag{2.6}$$

where β is a positive constant. This includes the practically interesting case where $\mathsf{h} = 0$ outside a finite range of x.†

The boundary condition that the normal component of velocity be continuous can be written

$$-\partial_x \psi = \epsilon\{\partial_t \mathsf{h} + (1+\partial_z \psi)\,\partial_x \mathsf{h}\} \quad \text{at} \quad z = \epsilon \mathsf{h}. \tag{2.7a}$$

If there is a horizontal upper boundary, we impose

$$\partial_x \psi = 0 \quad \text{at} \quad z = \pi. \tag{2.7b}$$

Also

$$\partial_x \psi, \quad \theta \to 0 \quad \text{as} \quad |x| \to \infty \quad (t < \infty). \tag{2.7c}$$

2.2. *Basic expansion in powers of amplitude*

It is now assumed that the dependent variables ψ and θ can be expanded in powers of ϵ, as follows:

$$\left.\begin{aligned} \psi(x,z,t;\,\epsilon) &= \epsilon\psi^{(1)}(x,z,t) + \epsilon^2\psi^{(2)}(x,z,t) + \ldots, \\ \theta(x,z,t;\,\epsilon) &= \epsilon\theta^{(1)}(x,z,t) + \epsilon^2\theta^{(2)}(x,z,t) + \ldots. \end{aligned}\right\} \tag{2.8}$$

These expansions are substituted into the equations and boundary conditions, and like powers of ϵ equated. To do this in (2.7*a*) we assume that ψ can be expanded in powers of z about $z = 0$ for each t and x such that $\mathsf{h} \neq 0$. The first-order problem in the bounded case is

$$\begin{cases} (\partial_t + \partial_x)\,\eta^{(1)} + \partial_x \theta^{(1)} = 0, & (2.9a) \\ (\partial_t + \partial_x)\,\theta^{(1)} - K^2\,\partial_x \psi^{(1)} = 0, & (2.9b) \\ \psi^{(1)} = \theta^{(1)} = 0 \quad \text{for} \quad t < 0, & (2.9c) \\ -\partial_x \psi^{(1)}(x,0,t) = (\partial_t + \partial_x)\,\mathsf{h}, & (2.9d) \\ \qquad \psi^{(1)}(x,\pi,t) = 0, & (2.9e) \\ \partial_x \psi^{(1)}, \quad \theta^{(1)} \to 0 \quad \text{as} \quad |x| \to \infty \quad (t < \infty). & (2.9f) \end{cases}$$

By definition $\eta^{(1)} = \nabla^2 \psi^{(1)}$, etc; also (2.7*b*) has been integrated with respect to x, with a convenient choice of the 'constant' of integration.

The second-order problem is

$$\begin{cases} (\partial_t + \partial_x)\,\eta^{(2)} + \partial_x \theta^{(2)} = \mathscr{A}, & (2.10a) \\ (\partial_t + \partial_x)\,\theta^{(2)} - K^2\,\partial_x \psi^{(2)} = \mathscr{B}, & (2.10b) \\ \psi^{(2)} = \theta^{(2)} = 0 \quad \text{for} \quad t < 0, & (2.10c) \\ \partial_x \psi^{(2)}(x,0,t) = \partial_x \hbar(x,t), & (2.10d) \\ \qquad \psi^{(2)}(x,\pi,t) = 0, & (2.10e) \\ \partial_x \psi^{(2)}, \quad \theta^{(2)} \to 0 \quad \text{as} \quad |x| \to \infty \quad (t < \infty). & (2.10f) \end{cases}$$

In the boundary condition (2.10*d*)

$$\hbar(x,t) \equiv -\mathsf{h}(x,t)\,\partial_z \psi^{(1)}(x,0,t), \tag{2.11}$$

† To forestall possible confusion throughout this paper, note that '$a = O(b)$' is not synonymous with a verbal expression like 'a is of order b', but includes the possibility that $a = o(b)$, i.e. that a/b vanishes in the limit (Lighthill 1958, p. 2).

a known function once (2.9) is solved. The role of $\partial_x \hat{h}$ as a forcing function in problem (2.10) is similar to the role of the right-hand side of (2.9*d*) in problem (2.9). In problem (2.10) there are two more forcing terms. The mechanical forcing $\mathscr{M}$ has the nature of the curl of a body force, while $\mathscr{B}$ represents a rate of input of buoyancy into, or rate of heating of, the interior of the fluid. $\mathscr{M}$ and $\mathscr{B}$ are defined by

$$\left.\begin{aligned}\mathscr{M}(x,z,t) &\equiv \partial(\psi^{(1)},\eta^{(1)})/\partial(x,z),\\ \mathscr{B}(x,z,t) &\equiv \partial(\psi^{(1)},\theta^{(1)})/\partial(x,z).\end{aligned}\right\} \tag{2.12}$$

They represent what the fluid feels as a result of redistribution of momentum and buoyancy by the first-order motions.

Throughout this paper we shall *assume* that the formal procedure leading to (2.9)–(2.12) is mathematically justifiable in some sense. The conditions under which this assumption seems likely to be true are discussed in §7; they are more restrictive than appears to have been commonly recognized in connection with discussions of Long's hypothesis.

2.3. *Definition of 'columnar disturbance'*

In the context of the time-dependent problem that is our main concern except in §6, the term 'columnar disturbance' will be understood to mean a contribution to ψ or θ which, for a general fixed value of z, is *one-signed throughout an x-region whose length continually increases as t, for large t.* The usual way in which such a disturbance arises is that one end of the expanding region propagates freely with the long-wave speed $c_n(0)$ (like a change of surface elevation in free-surface channel flow), whilst the other end is supported by continued forcing which itself is confined to some comparatively small region.

The kind of forcing that will support columnar disturbances can appropriately be called forcing 'of type one'; what this involves will become apparent from a simple physical argument given in §4.4. Qualitatively speaking, a central result of §4 is that the contributions to $\mathscr{M}$ and $\mathscr{B}$ due to the self-interaction of a lee-wave tail do, in fact, represent forcing of type one (in complete contrast with the type-zero forcing represented by the right-hand side of (2.9*d*)).

3. First-order solution for the inviscid, bounded problem

Throughout §§3, 4 and 5 it will be assumed that

$$1 < K \neq \text{integer}. \tag{3.1}$$

The largest integer less than K will be denoted by n_K.

To fix ideas, attention will be restricted to the particular case

$$\mathsf{h}(x,t) = \left\{\begin{matrix} 0 & \text{for} & t \leqslant 0, \\ h(x) & \text{for} & t > 0. \end{matrix}\right\} \tag{3.2}$$

It is emphasized, however, that the leading results concerning columnar-disturbance generation can be shown (§4.6) to be unaltered by introducing more general ways for $\mathsf{h}(x,t)$ to increase from zero with time.

3.1. *Solution of problem* (2.9) *by integral transforms*

We define Fourier-transformed dependent variables (in which there will be no need to retain the superscript (1)) as follows:

$$\tilde{\psi}(k,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi^{(1)}(x,z,t)\, e^{-ikx}\, dx, \tag{3.3}$$

and similarly $\tilde{\theta}(k,z,t)$. Also

$$\tilde{\mathrm{h}}(k,t) = \begin{cases} 0 & \text{for} \quad t \leqslant 0, \\ \tilde{h}(k) & \text{for} \quad t > 0, \end{cases}$$

where
$$\tilde{h}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(x)\, e^{-ikx}\, dx. \tag{3.4}$$

By (2.6), $\tilde{h}(k)$ is regular in a strip

$$|\mathrm{Im}\,(k)| < \beta, \tag{3.5a}$$

and it is convenient to assume that $h(x)$ is sufficiently well-behaved that

$$\left.\begin{aligned} \tilde{h}(k) &= O(|k|^{-1-\delta}) \\ \partial_k \tilde{h} &= O(|k|^{-2}) \\ \partial_k^2 \tilde{h} &= O(|k|^{-3}) \end{aligned}\right\} \quad \text{uniformly for } |\mathrm{Im}\,(k)| < \beta, \tag{3.5b}$$

as $|k| \to \infty$, where δ is some positive constant.

After Fourier and Laplace-transforming problem (2.9) and solving, we may carry out the Laplace inversion exactly since it is found that the only singularities are poles on the imaginary axis, corresponding to the real frequencies

$$\omega = 0,$$
$$\omega = k(1 \pm c_n) \quad (n = 1, 2, 3, \ldots),$$

where
$$c_n = c_n(k) \equiv K(k^2+n^2)^{-\frac{1}{2}} \quad (>0, k\,\text{real}). \tag{3.6}$$

For $t > 0$, i.e. after the initial, irrotational response of the fluid to the introduction of the obstacle, the result for $\tilde{\psi}$ can be written

$$\tilde{\psi}(k,z,t) = \tilde{\psi}_S(k,z) + \tilde{\psi}^+(k,z,t) + \tilde{\psi}^-(k,z,t) \quad (t > 0), \tag{3.7}$$

where the first, steady, contribution arises from the pole at the origin and is found to be

$$\tilde{\psi}_S(k,z) = -\tilde{h}(k)\frac{\sin\{(K^2-k^2)^{\frac{1}{2}}(\pi - z)\}}{\sin\{(K^2-k^2)^{\frac{1}{2}}\pi\}}, \tag{3.8}$$

with the same choice of branch in the numerator as in the denominator. The remaining terms are given by

$$\tilde{\psi}^\pm(k,z,t) = \sum_{n=1}^{\infty} \tilde{\psi}_n^\pm(k,t)\sin(nz), \tag{3.9}$$

where
$$\tilde{\psi}_n^\pm(k,t) \equiv \frac{nc_n^4\tilde{h}(k)\exp[-ik(1\pm c_n)t]}{\pi K^2(1\pm c_n)}. \tag{3.10}$$

Similarly,
$$\tilde{\theta}(k,z,t) = \tilde{\theta}_S(k,z)+\tilde{\theta}^+(k,z,t)+\tilde{\theta}^-(k,z,t) \quad (t>0), \tag{3.11}$$
where
$$\tilde{\theta}^\pm = \sum_{n=1}^{\infty} \tilde{\theta}_n^\pm \sin(nz) \tag{3.12}$$
and
$$\tilde{\theta}_S \equiv K^2\tilde{\psi}_S, \quad \tilde{\theta}_n^\pm \equiv \mp K^2 c_n^{-1}\tilde{\psi}_n^\pm. \tag{3.13}$$

We note that $c_n(k)$ is the intrinsic phase speed of the nth mode at wavenumber k, and that the 'plus' and 'minus' contributions to $\tilde{\psi}$ and $\tilde{\theta}$ are associated with waves whose phases propagate respectively with, and against, the basic flow.

It will be observed that the separate contributions $\tilde{\psi}^-$ and $\tilde{\theta}^-$ have poles on the positive real k-axis, at each of the finite number of points for which $c_n = 1$, viz.
$$k = \mathfrak{k}_n \equiv (K^2-n^2)^{\frac{1}{2}} > 0 \quad (1 \leqslant n \leqslant n_K). \tag{3.14}$$
Since $K^2 - \mathfrak{k}_n^2 = n^2$, (3.8) and (3.13) show that these are also the poles of $\tilde{\psi}_S$ and $\tilde{\theta}_S$. A simple calculation shows that the residues for $\tilde{\psi}_S$ and $\tilde{\psi}^-$ are equal and opposite, as are those for $\tilde{\theta}_S$ and $\tilde{\theta}^-$. It follows that $\tilde{\psi}$ and $\tilde{\theta}$ are regular, in a strip
$$|\text{Im}(k)| \leqslant \beta',$$
where the positive constant
$$\beta' < \min[\beta, \{(n_K+1)^2 - K^2\}^{\frac{1}{2}}]. \tag{3.15}$$

Further, since $h(x)$ is real, $\tilde{h}(-k) = \tilde{h}^*(k)$ for real k; (3.6)–(3.13) then imply that the same is true for $\tilde{\psi}$ and $\tilde{\theta}$. Similarly, $\tilde{\psi}dk$ and $\tilde{\theta}dk$ are pure imaginary if k and dk are pure imaginary and if k lies between $\pm i\beta'$. Therefore the inverse Fourier transforms of $\tilde{\psi}$ and $\tilde{\theta}$, giving the desired solution to problem (2.9), can be written not only as integrals from $-\infty$ to $+\infty$ but also as
$$\psi^{(1)}(x,z,t) = 2\,\text{Re}\int_{i\mu}^{\infty} \tilde{\psi} e^{ikx} dk, \quad \theta^{(1)} = 2\,\text{Re}\int_{i\mu}^{\infty} \tilde{\theta} e^{ikx} dk \quad (-\beta' < \mu < \beta'), \tag{3.16}$$
where the path is to lie within $|\text{Im}(k)| \leqslant \beta'$. The path may be taken on, or to either side of, the real axis (with an appropriate choice of the real number μ) in virtue of the regularity of $\tilde{\psi}$ and $\tilde{\theta}$.

3.2. *The tail of the nth lee-wave train*

The magnitude of the intrinsic group velocity of a component whose intrinsic phase speed is $c_n(k)$ is
$$\gamma_n(k) = |\partial_k(kc_n)| = Kn^2(k^2+n^2)^{-\frac{3}{2}} > 0. \tag{3.17}$$
In particular, it will be found that the tail of the nth lee-wave train moves downstream with absolute velocity
$$\begin{aligned} \mathfrak{v}_n &\equiv 1-\gamma_n(\mathfrak{k}_n), \\ &= 1-\frac{n^2}{K^2} = \frac{\mathfrak{k}_n^2}{K^2}. \end{aligned} \tag{3.18}$$

We anticipate that the contributions to $\psi^{(1)}(x,z,t)$ containing lee waves are those for which the corresponding separate contributions to $\tilde{\psi}$ have poles at $k = \mathfrak{k}_n$, namely

$$\psi_S(x,z) + \sum_{n=1}^{n_K} \psi_n^-(x,t)\sin(nz), \tag{3.19a}$$

where

$$\psi_S(x,z) \equiv 2\,\mathrm{Re}\int_{i\mu}^{\infty} \tilde{\psi}_S(k,z)\, e^{ikx}\, dk \tag{3.19b}$$

and (cf. (3.10))

$$\psi_n^{\pm}(x,t) \equiv 2\,\mathrm{Re}\int_{i\mu}^{\infty} \frac{nc_n^4(k)\tilde{h}(k)\exp[-ik(1\pm c_n)t + ikx]}{\pi K^2(1\pm c_n)}\, dk. \tag{3.19c}$$

The paths of integration may be taken above or below the pole at $k = \mathfrak{k}_n$, provided that the same is done for each ψ_n^- as is done for ψ_S. From now on each path will be taken above $k = \mathfrak{k}_n$, unless otherwise stated.

It is shown in § 3.3 that this choice makes the steady part ψ_S exponentially small as $x \to +\infty$, i.e. far downstream. This means that ψ_S will not by itself describe the steady lee waves at $t = \infty$, in the usual way. But ψ_n^-, by itself, will describe the tail of the nth lee-wave train.

We can rearrange ψ_n^- (without approximation) in the form

$$\psi_n^- = \mathrm{Re}\{\mathfrak{F}_n(x,t)\exp(i\mathfrak{k}_n x)\}; \tag{3.20}$$

by definition

$$\mathfrak{F}_n(x,t) \equiv \int_{\substack{i\mu \\ [\text{above } k=\mathfrak{k}_n]}}^{\infty} \mathfrak{f}_n(k)\exp\{itk(c_n-1) + i(k-\mathfrak{k}_n)x\}\frac{dk}{k-\mathfrak{k}_n}, \tag{3.21}$$

where the function $\mathfrak{f}_n$ in the integrand is defined by

$$\mathfrak{f}_n(k) \equiv \frac{2n}{\pi K^2}\frac{c_n^4(k)\,\tilde{h}(k)}{\{1-c_n(k)\}}(k-\mathfrak{k}_n). \tag{3.22}$$

Note that $\mathfrak{f}_n$ is regular in the right-hand half of the strip $|\mathrm{Im}(k)| \leqslant \beta'$, and that

$$\mathfrak{f}_n = O(|k|^{-4-\delta})$$

as $|k| \to \infty$, uniformly in $|\mathrm{Im}(k)| \leqslant \beta'$, by (3.5b) and (3.6).

It will be convenient to change the variable of integration in (3.21) to

$$l \equiv t^{\frac{1}{2}}(k-\mathfrak{k}_n). \tag{3.23}$$

Writing also

$$\mathfrak{x}_n \equiv t^{-\frac{1}{2}}(x - \mathfrak{v}_n t), \tag{3.24}$$

we find for the exponent in (3.21)

$$\begin{aligned}\{\;\} &= itk[c_n(k)-1] + i(k-\mathfrak{k}_n)(\mathfrak{v}_n t + t^{\frac{1}{2}}\mathfrak{x}_n)\\ &= i\varphi_n(l,t) + i\,l\mathfrak{x}_n,\end{aligned}$$

say, where

$$\varphi_n(l,t) \equiv \tfrac{1}{2}\gamma_n'(\mathfrak{k}_n)\,l^2 + \tfrac{1}{6}t^{-\frac{1}{2}}\gamma_n''(\mathfrak{k}_n)\,l^3 + \dots, \tag{3.25}$$

and the coefficients

$$\{\gamma_n'(\mathfrak{k}_n), \gamma_n''(\mathfrak{k}_n), \dots\} = \{\partial_k\gamma_n(k), \partial_k^2\gamma_n(k), \dots\}|_{k=\mathfrak{k}_n}.$$

$\mathfrak{F}_n$ can now be rewritten as

$$\mathfrak{F}_n(x,t) \equiv \int_{\substack{t^{\frac{1}{2}}(-\mathfrak{k}_n+i\mu) \\ [\text{above } l=0]}}^{\infty} e^{il\mathfrak{x}_n}\,\mathfrak{f}_n(\mathfrak{k}_n + t^{-\frac{1}{2}}l)\exp\{i\varphi_n(l,t)\}\frac{dl}{l}. \tag{3.26}$$

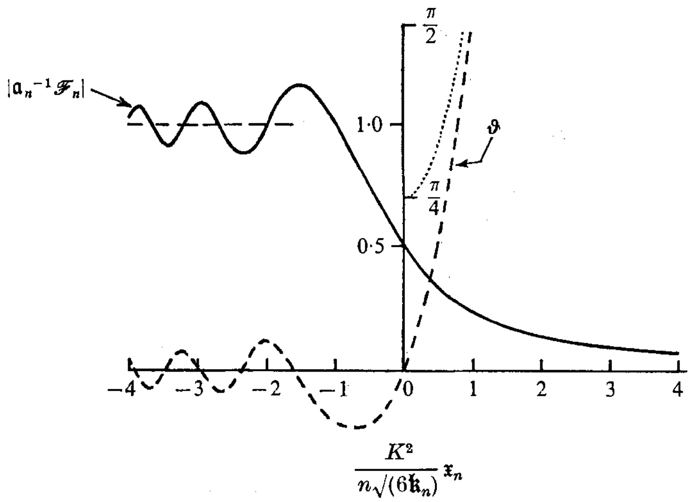

FIGURE 1. Limiting form of the complex wave amplitude in the lee-wave tail. The solid curve, taken with the left-hand ordinate scale, represents the modulus of $\mathfrak{a}_n^{-1}\mathscr{F}_n$; the broken curve, with the right-hand ordinate scale, represents its phase ϑ ($\mathfrak{a}_n^{-1}\mathscr{F}_n = |\mathfrak{a}_n^{-1}\mathscr{F}_n|e^{i\vartheta}$). The fine dotted curve represents the asymptote $\vartheta \sim \{\frac{1}{4}\pi + (\text{abscissa})^2\}$ (Abramowitz & Stegun 1964, eqs. (7.3.9, 10, 22, 27, 28).

To help visualize the qualitative appearance of $\mathfrak{F}_n$, we note the leading saddle-point approximation to it, as $t \to \infty$ with $\mathfrak{x}_n = O(1)$. This approximation has error $O(t^{-\frac{1}{2}})$, and is the result of neglecting all terms that are $O(t^{-\frac{1}{2}})$ in (3.26), and replacing the lower limit of integration by $-\infty$, i.e.

$$\mathfrak{F}_n \sim \mathscr{F}_n(\mathfrak{x}_n) \equiv \mathfrak{f}_n(\mathfrak{k}_n)\int_{\substack{-\infty \\ [\text{above}]}}^{\infty} \exp\{\tfrac{1}{2}i\gamma_n'(\mathfrak{k}_n)\,l^2 + i\mathfrak{x}_n l\}\frac{dl}{l}$$
$$= \mathfrak{f}_n(\mathfrak{k}_n)\int_{\mathfrak{x}_n}^{\infty} d\mathfrak{x}'\int_{-\infty}^{\infty} -i\exp\{\tfrac{1}{2}i\gamma_n'(\mathfrak{k}_n)\,l^2 + i\mathfrak{x}'l\}\,dl.$$

Carrying out the second integration we find that

$$\mathscr{F}_n(\mathfrak{x}_n) = \tfrac{1}{2}\mathfrak{a}_n \operatorname{erfc}\left\{\frac{K^2 e^{-\frac{1}{4}i\pi}\,\mathfrak{x}_n}{n\surd(6\mathfrak{k}_n)}\right\}, \tag{3.27}$$

where the complex constant

$$\mathfrak{a}_n = -4i\mathfrak{k}_n^{-1}n\tilde{h}(\mathfrak{k}_n). \tag{3.28}$$

Graphs of the modulus and phase of $\mathscr{F}_n(\mathfrak{x}_n)$ are given in figure 1; $\mathscr{F}_n$ has the form of the complex Fresnel amplitude characteristic of the general asymptotic behaviour of the envelope of a dispersing semi-infinite sinusoidal wave. Since $\mathscr{F}_n$ approaches $\mathfrak{a}_n$ as $\mathfrak{x}_n$ decreases, we anticipate that $\mathfrak{a}_n$ is the complex amplitude of the stream function for the nth lee-wave train.

The foregoing approximate representation (3.27) of $\mathscr{F}_n$ helps to fix ideas, but we emphasize that it is not used in the subsequent analysis, which will make use only of a few rather general properties of $\mathfrak{F}_n$ (some of which cannot, in any case, be deduced from (3.27)). The needed properties are that, provided $|\mathfrak{x}_n|$ is not too large, say

$$\mathfrak{x}_n = o(t^{\frac{1}{2}}), \tag{3.29}$$

then (a)

$$\partial_t \mathfrak{F}_n = -\mathfrak{v}_n \partial_x \mathfrak{F}_n + O(t^{-1}\hat{\mathfrak{x}}_n) \tag{3.30a}$$

where $\hat{\mathfrak{x}}_n \equiv \max(|\mathfrak{x}_n|, 1)$; and (b)

$$\mathfrak{F}_n = \{\mathfrak{a}_n + O(\hat{\mathfrak{x}}_n^{-1})\} \quad \text{or} \quad O(\hat{\mathfrak{x}}_n^{-1}) \quad (\mathfrak{x} \lessgtr 0); \tag{3.30b}$$

and (c) $\mathfrak{F}_n$ is smooth enough to satisfy

$$\partial_x^p \partial_t^{r-p} \mathfrak{F}_n = O(t^{-\frac{1}{2}r}\hat{\mathfrak{x}}_n^{r-1}) \quad (1 \leqslant r \leqslant 4;\ p \leqslant r). \tag{3.30c}$$

These relations follow easily from the exact expression (3.21) or (3.26), after first carrying out any differentiations required and then estimating the resulting integrals.†

From (3.13), the contribution to $\theta^{(1)}$ corresponding to $\psi_n^- \sin(nz)$ is, say,

$$\theta_n^- \sin(nz) \equiv \operatorname{Re}\{\mathfrak{G}_n(x,t)\exp(i\mathfrak{k}_n x)\}\sin(nz), \tag{3.31a}$$

where $\mathfrak{G}_n$ is the same as (3.26) apart from insertion of the following extra factor into the integrand:

$$K^2\{c_n(\mathfrak{k}_n + t^{-\frac{1}{2}}l)\}^{-1}, \tag{3.31b}$$

which can be expanded as

$$K^2 + \mathfrak{k}_n t^{-\frac{1}{2}} l + O(t^{-1}l^2).$$

Now

$$\mathfrak{k}_n t^{-\frac{1}{2}} l\, e^{il\mathfrak{x}_n} = -i\mathfrak{k}_n \partial_x e^{il\mathfrak{x}_n},$$

and so we have (still assuming $\mathfrak{x}_n = o(t^{\frac{1}{2}})$)

$$\mathfrak{G}_n - K^2\mathfrak{F}_n = -i\mathfrak{k}_n \partial_x \mathfrak{F}_n + O(t^{-1}\hat{\mathfrak{x}}_n), \tag{3.32}$$

a relation that will be needed for evaluating the nonlinear term $\mathscr{B}$ in §4.

It can also be shown in the same way as for (3.30) that, consistently with (3.32),

$$\partial_x^p \partial_t^{r-p}(\mathfrak{G}_n - K^2\mathfrak{F}_n) = O(t^{-\frac{1}{2}(r+1)}\hat{\mathfrak{x}}_n^r) \quad (0 \leqslant r \leqslant 3;\ p \leqslant r), \tag{3.33}$$

provided that $\mathfrak{x}_n = o(t^{\frac{1}{2}})$.

The results (3.30)–(3.33) contain all the information that is directly made use of in §4.

3.3. *The 'purely transient' and the steady contributions*

The representations (3.20) and (3.31a) are useful only in the tail region $x \sim \mathfrak{v}_n t$. Outside it, and for those contributions $\psi_n^{\pm}$ that do not contain any lee waves, different forms are more useful. These are used in obtaining the results of §5, and show in particular that all contributions to $\psi^{(1)}$ apart from the lee waves and their tails, and the steady obstacle near-field, are evanescent, algebraically or faster, as $t \to \infty$. The latter conclusion also follows, of course, from the standard

† Taking a path of integration similar to that shown in figure 2 below, one notes that in each case (3.29) makes the first term in (3.25) dominant near the saddle point $l = -\mathfrak{x}_n/\gamma_n'(\mathfrak{k}_n)$. The hypothesis (3.5b) permits an easy proof that there is no significant contribution from large $|k|$; the factors multiplying the exponentials in the integrands are absolutely integrable, being $O(k^{r-5-\delta})$ uniformly over the strip $|\operatorname{Im}(k)| \leqslant \beta'$ as $|k| \to \infty$.

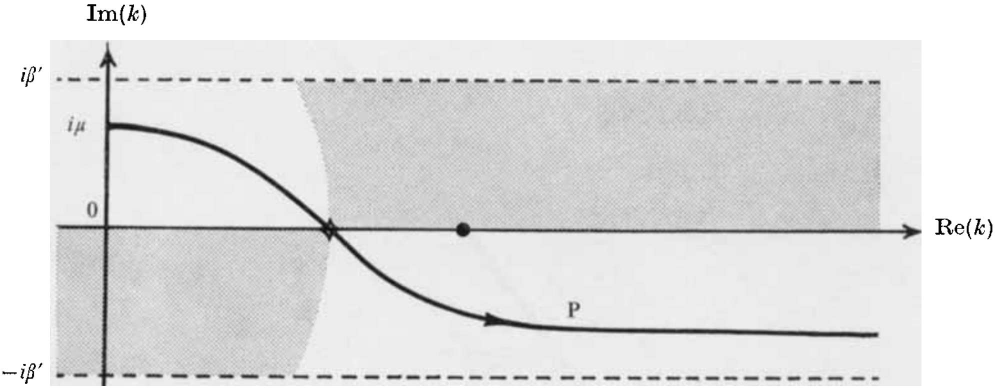

FIGURE 2. Path of integration, P, for (3.37*b*), when $k_{0n} < \mathfrak{k}_n$, i.e. when $x < \mathfrak{v}_n t$, upstream of the tail. The dot indicates the pole at $k = \mathfrak{k}_n$; the path crosses the real axis at the saddle point $k = k_{0n}$.

saddle-point approximation; but for the purposes of § 5 this approximation must be avoided, as before.

The starting point is the following slight rearrangement of (3.19*c*):

$$\psi_n^{\pm}(x,t) = \psi_{0n}^{\pm}(x,t) + \mathrm{Re}\,\{L_n^{\pm}\exp(i\mathfrak{k}_n x)\}; \tag{3.34}$$

$$\psi_{0n}^{\pm}(x,t) = \mathrm{Re}\int_{\mathrm{P}} f_n^{\pm}(k)\exp[-ik(1\pm c_n)t + ikx]\,dk, \tag{3.35}$$

where the path P lies in $|\mathrm{Im}\,(k)| \leqslant \beta'$ and is chosen to make the integrand as small as possible consistent with this. We have defined

$$\begin{aligned} f_n^{\pm}(k) &\equiv \frac{2n}{\pi K^2}\frac{c_n^4(k)\,\tilde{h}(k)}{\{1\pm c_n(k)\}} \\ &= O(|k|^{-5-\delta}) \quad (|k| \to \infty, |\mathrm{Im}\,(k)| \leqslant \beta'), \end{aligned}$$

$$L_n^- = \begin{cases} \mathfrak{a}_n & (x < \mathfrak{v}_n t \quad \text{and} \quad n \leqslant n_K), \qquad (3.36a) \\ 0 & (x > \mathfrak{v}_n t \quad \text{or} \quad n > n_K), \qquad (3.36b) \end{cases}$$

$$L_n^+ = 0 \qquad (\text{all } x,\, n). \tag{3.36c}$$

For an observer moving at constant velocity, either $\psi_{0n}^{\pm}$ is exponentially small as $t \to \infty$, or there is a saddle point on $0 \leqslant k \leqslant \infty$, in which case the path P must be taken through it as in figure 2, keeping to the unshaded regions (within which the integrand is exponentially small). In the case of ψ_n^- $(n \leqslant n_K)$ there is also a pole on the positive real axis. In figure 2 the path P is shown running below the pole, corresponding to the possibility (3.36*a*), in which $(\psi_{0n}^- - \psi_n^-) = \mathrm{Re}$ ($2\pi i$ times the residue at the pole).

The regions of space-time in which ψ_{0n}^- is not exponentially small as $t \to \infty$ are illustrated for the case $n \leqslant n_K$ as regions i to vii in figure 3. Those in which ψ_{0n}^+ is not exponentially small are regions vii to ix. In such regions it is useful to rewrite (3.35) and its derivatives with respect to x in the form

$$\partial_x^r\,\psi_{0n}^{\pm} = \mathrm{Re}\,\{F_{n,r}^{\pm}\exp(ik_{0n}x - i\omega_{0n}t)\} \tag{3.37a}$$

where

$$F_{n,r}^{\pm}(x,t) \equiv \int_{\mathrm{P}} (ik)^r f_n^{\pm}(k)\exp\{it\phi_n^{\pm}(k;\,x,t)\}\,dk. \tag{3.37b}$$

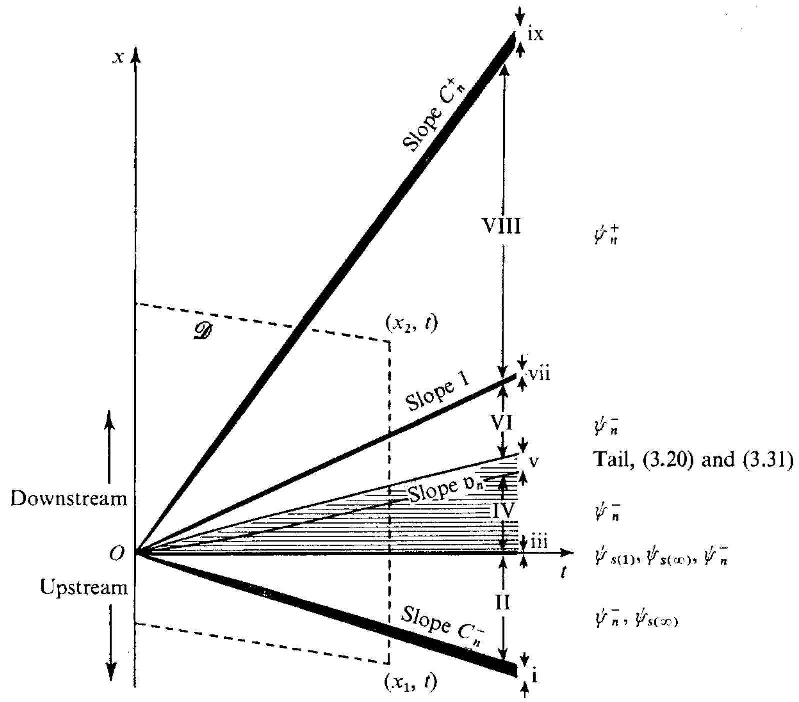

FIGURE 3. Regions for describing the asymptotic behaviour as $t \to \infty$ of the $\sin nz$ component of the first-order solution $\psi^{(1)}(x, z, t)$, for $n \leqslant n_K$. Regions i and ix are of order $t^{\frac{1}{3}}$ in size, region iii is of order unity in size, region v, the tail, is of order $t^{\frac{1}{2}}$ (it corresponds to $|\mathfrak{x}_n| \lesssim 1$, in the notation of § 3.2), and region vii is of order $t^{\frac{1}{4}}$. The remaining regions, denoted by the upper case Roman numerals, are of order t in size. The corresponding diagram for $n > n_K$ is similar except that regions iii and v are absent, i is downstream of $x = 0$, and II, IV and VI merge together.

The parallelogram $\mathscr{D}$ is referred to in § 5.

By choosing k_{0n} and ω_{0n} as the wavenumber and frequency at which the absolute group velocity is xt^{-1}, corresponding to the saddle point, the function $\phi_n^{\pm}$ can be made to have the Taylor expansion

$$\phi_n^{\pm}(k;\, x, t) = \mp \{\tfrac{1}{2}\gamma_n'(k_{0n})\,(k-k_{0n})^2 + \tfrac{1}{6}\gamma_n''(k_{0n})\,(k-k_{0n})^3 + \ldots\}. \qquad (3.37c)$$

The dependence of k_{0n} and ω_{0n} upon x and t, slowly varying for large t, is given explicitly by

$$k_{0n}(x,t) \equiv n\left[\left(\frac{C_n}{X}\right)^{\frac{2}{3}} - 1\right]^{\frac{1}{2}} \quad (>0, \quad |X| < C_n), \qquad (3.38a)$$

where

$$X \equiv xt^{-1} - 1 \qquad (3.38b)$$

and

$$C_n \equiv c_n(0) = K/n > 0;$$

also

$$\begin{aligned} \omega_{0n}(x,t) &\equiv k_{0n}\{1 \pm c_n(k_{0n})\} \\ &= k_{0n}\{1 + (C_n^2 X)^{\frac{1}{3}}\} \quad (\operatorname{sgn}(\;)^{\frac{1}{3}} = \operatorname{sgn} X). \end{aligned} \qquad (3.38c)$$

$F_{n,r}^{\pm}(x,t)$ is found to be similarly slowly varying, in a sense that can be made precise by forming its derivatives with respect to x and t and then estimating the result in the same way as led to (3.30). $F^{\pm}_{n,r}$ itself is generally $O(t^{-\frac{1}{2}})$ in magnitude; but care is needed in order to take correct account of details near the tail region v if present, near regions i and ix where k_{0n} is small, and near region vii where it is large.† It is upon estimates of $F^{\pm}_{n,r}$ and its derivatives, and corresponding estimates concerning $\theta^{\pm}_{n}$, that much of the error analysis of §5 is based.

There remains the time-independent contribution to $\psi^{(1)}$ given by (3.19b), in which $\tilde{\psi}_{\mathrm{s}}$ is defined by (3.8). We note that $\tilde{\psi}_{\mathrm{s}} = -\tilde{h}$ for $z = 0$, and restrict attention to fixed $z > 0$ (and $\leqslant \pi$).

Now (3.8) can be written

$$\tilde{\psi}_{\mathrm{s}}(k,z) = \tilde{\psi}_{\mathrm{s}(1)} + \tilde{\psi}_{\mathrm{s}(\infty)} \tag{3.39}$$

where the regular part

$$\tilde{\psi}_{\mathrm{s}(1)} = -\tilde{h}(k)\frac{\sin\{(K^2-k^2)^{\frac{1}{2}}(\pi - z)\}}{\sin\{(K^2-k^2)^{\frac{1}{2}}\pi\}} - \tilde{\psi}_{\mathrm{s}(\infty)}; \tag{3.40a}$$

$$\tilde{\psi}_{\mathrm{s}(\infty)} = -\sum_{n=1}^{n_K}\frac{n}{\pi \mathfrak{k}_n}\left[\frac{\tilde{h}(\mathfrak{k}_n)}{k-\mathfrak{k}_n} - \frac{\tilde{h}(-\mathfrak{k}_n)}{k+\mathfrak{k}_n}\right]\sin(nz). \tag{3.40b}$$

It can be shown from (3.5) (with reference to Lighthill (1958, p. 49)) that

$$\psi_{\mathrm{s}(1)} \equiv 2\,\mathrm{Re}\int_{i\mu}^{\infty}\tilde{\psi}_{\mathrm{s}(1)}e^{ikx}\,dk = o(e^{-\beta'|x|}) \quad \text{as} \quad |x| \to \infty, \tag{3.41a}$$

where β' is the positive constant defined by (3.15). The corresponding contribution

$$\theta_{\mathrm{s}(1)} = K^2\psi_{\mathrm{s}(1)}. \tag{3.41b}$$

Therefore $\psi_{\mathrm{s}(1)}$ and $\theta_{\mathrm{s}(1)}$ represent only 'near-field' contributions, exponentially small at large distances from the obstacle. Also

$$\psi_{\mathrm{s}(\infty)} \equiv 2\,\mathrm{Re}\underset{\text{[above]}}{\int_{i\mu}^{\infty}}\tilde{\psi}_{\mathrm{s}(\infty)}e^{ikx}\,dk = \underset{\text{[above]}}{\int_{-\infty}^{\infty}}\tilde{\psi}_{\mathrm{s}(\infty)}e^{ikx}\,dk,$$

which upon substitution from (3.40b) can be evaluated as a sum of residues. Denoting Heaviside's step function by $H(x)$, we have

$$\psi_{\mathrm{s}(\infty)} = -H(-x)\sum_{n=1}^{n_K}\mathrm{Re}\,(\mathfrak{a}_n e^{i\mathfrak{k}_n x})\sin(nz) \quad (= K^{-2}\theta_{\mathrm{s}(\infty)}), \tag{3.42}$$

where $\mathfrak{a}_n = -4i\mathfrak{k}_n^{-1}n\tilde{h}(\mathfrak{k}_n)$, as before. (Of course, the discontinuity at $x = 0$ must be cancelled by an opposite jump in $\psi_{\mathrm{s}(1)}$. The sum $\psi_{\mathrm{s}} = \psi_{\mathrm{s}(1)} + \psi_{\mathrm{s}(\infty)}$ must, indeed, have a very smooth x-dependence (for $0 < z \leqslant \pi$), because of the exponential smallness of its Fourier transform $\tilde{\psi}_{\mathrm{s}} = O(k^{-1-\delta}e^{-z|k|})$ as $\mathrm{Re}(k) \to \infty$.)

The statement made in §3.2 that $\psi_{\mathrm{s}} \to 0$, exponentially as $x \to +\infty$, is now verified by (3.41) and (3.42). For $x > 0$, (3.42) is zero and as soon as we are far enough downstream for the near field, (3.41), to be negligible, the lee waves are

† It is for the latter region that it is convenient to assume the second and third of (3.5b). More detail is given in longer versions of §§3.3 and 5, copies of which are available on request from the *J.F.M.* editorial office, D.A.M.T.P., Silver St, Cambridge.

described entirely by the last term of (3.34), or, near $x = \mathfrak{v}_n t$, by (3.20). For $x < 0$, the last term in (3.34) times $\sin(nz)$ precisely cancels the corresponding term in (3.42), so that the near-field $\psi_{s(1)}$ is the only steady contribution for $x < 0$.

The steady part of the drag on the obstacle per unit spanwise distance is found to be

$$\tfrac{1}{4}\pi \sum_{n=1}^{nK} \mathfrak{k}_n^2 |\mathfrak{a}_n|^2 \tag{3.43}$$

in our notation. (To dimensionalize this, multiply by $U^2 D \times$ (density), remembering that D is π^{-1} times the channel height.)

In summary, the steady part of the complete solution is, say,

$$\psi_s' = K^{-2}\theta_s' = \psi_{s(1)} + \psi_{s(\infty)} + \sum_{n=1}^{nK} \{\text{last term of (3.34)}^-\} \sin(nz). \tag{3.44}$$

4. The second-order columnar disturbances in the inviscid bounded problem

4.1. *The nonlinear forcing due to the lee-wave tails*

In the problem (2.10) for $\psi^{(2)}$ and $\theta^{(2)}$, the inhomogeneous forcing functions $\mathscr{M}(x,z,t)$, $\mathscr{B}(x,z,t)$, $\hbar(x,t)$ that appear on the right-hand sides of (2.10*a, b, d*) are now known functions because we know $\psi^{(1)}$ and $\theta^{(1)}$.

In evaluating $\mathscr{M}$ and $\mathscr{B}$ it will be convenient to regard each of them as sums of various separate contributions. Since (2.10) is a linear problem, we may solve separately for the corresponding contributions to $\psi^{(2)}, \theta^{(2)}$, and likewise for the contribution to $\psi^{(2)}$ and $\theta^{(2)}$ due to the boundary-condition inhomogeneity $\hbar$.

In order to reach the main results as quickly as possible we restrict attention, in §4, to certain contributions to $\psi^{(2)}$ and $\theta^{(2)}$ which will be denoted by

$$\Psi(x,z,t), \quad \Theta(x,z,t),$$

and which will turn out to contain the dominant columnar disturbances. Ψ and Θ correspond to those parts of $\mathscr{M}$ and $\mathscr{B}$ that are associated with the self-interactions of the lee-wave tails. It can be shown (§5) that all other contributions to $\mathscr{M}$ and $\mathscr{B}$ give rise to columnar disturbances of relatively insignificant strength, if any, and that the same applies to $\hbar$.

The problem to be considered in this section, then, takes the form

$$\begin{cases} (\partial_t + \partial_x)\nabla^2\Psi + \partial_x\Theta = \sum_{n=1}^{nK} \mathfrak{M}_{nn}(x,t)\sin(2nz), & (4.1a) \\ (\partial_t + \partial_x)\Theta - K^2\partial_x\Psi = \sum_{n=1}^{nK} \mathfrak{B}_{nn}(x,t)\sin(2nz), & (4.1b) \\ \Psi = \Theta = 0 \quad \text{for} \quad t < 0, & (4.1c) \\ \Psi(x,0,t) = 0, & (4.1d) \\ \Psi(x,\pi,t) = 0, & (4.1e) \\ \partial_x\Psi, \quad \Theta \to 0 \quad \text{as} \quad |x| \to \infty \quad (t < \infty). & (4.1f) \end{cases}$$

To define the right-hand sides, let

$$\mathfrak{e}_{nn} = 0 \begin{cases} \text{for} \quad t < 0 \quad \text{and} \\ \text{for} \quad |\mathfrak{x}_n| > \mathfrak{X} \quad (\text{all } t) \end{cases} \quad [\mathfrak{x}_n \equiv t^{-\frac{1}{2}}(x - \mathfrak{v}_n t)] \tag{4.2a}$$

and $\mathfrak{e}_{nn} = 1$ otherwise, where $\mathfrak{X} > 0$ will be left unspecified for the present, except to say that

$$\mathfrak{X} = o(t^{\frac{1}{6}}). \tag{4.2b}$$

Now, noting (3.20) and (3.31a), define

$$\left.\begin{aligned} \mathfrak{M}_{nn} \sin(2nz) &= \mathfrak{e}_{nn} \frac{\partial[\sin(nz)\,\mathrm{Re}\,(\mathfrak{F}_n e^{i\mathfrak{k}_n x}), \quad \nabla^2\{\sin(nz)\,\mathrm{Re}\,(\mathfrak{F}_n e^{i\mathfrak{k}_n x})\}]}{\partial(x,z)}, \\ \mathfrak{B}_{nn} \sin(2nz) &= \mathfrak{e}_{nn} \frac{\partial[\sin(nz)\,\mathrm{Re}\,(\mathfrak{F}_n e^{i\mathfrak{k}_n x}), \quad \sin(nz)\,\mathrm{Re}\,(\mathfrak{G}_n e^{i\mathfrak{k}_n x})]}{\partial(x,z)}. \end{aligned}\right\} \tag{4.3}$$

In virtue of (4.2a) $\mathfrak{M}_{nn}$ and $\mathfrak{B}_{nn}$ represent a forcing region whose centre moves away from the origin with velocity $\mathfrak{v}_n$.

In what follows, the intrinsic long-wave speed

$$C_{2n} \equiv c_{2n}(0) = \gamma_{2n}(0) = K/2n, \tag{4.4a}$$

for the $2n$th mode, will play a central role. We define also the absolute long-wave velocities

$$\left.\begin{aligned} C_{2n}^{+} &\equiv 1 + C_{2n}, \quad > 1, \\ C_{2n}^{-} &\equiv 1 - C_{2n}, \quad \neq 0, \end{aligned}\right\} \tag{4.4b}$$

recalling (3.1).

4.2. *Calculation of* $\mathfrak{M}_{nn}$ *and* $\mathfrak{B}_{nn}$

For large t, advantage can be taken of the slowly varying properties (3.30c) etc. of $\mathfrak{F}_n(x,t)$ and $\mathfrak{G}_n(x,t)$ when carrying out the differentiations with respect to x in (4.3).

Let Φ stand for some function of x and t. Using (3.30c) we can show that

$$\frac{\partial[\sin(nz)\,\mathrm{Re}\,(\mathfrak{F}_n e^{i\mathfrak{k}_n x}), \quad \sin(nz)\,\mathrm{Re}\,(\Phi e^{i\mathfrak{k}_n x})]}{\partial(x,z)}$$

$$= \begin{cases} \tfrac{1}{2} n \mathfrak{k}_n \sin(2nz)\,\mathrm{Im}\,(\mathfrak{F}_n^* \Phi) + O(t^{-1}\hat{\mathfrak{x}}_n), & \\ \quad \text{if} \quad \Phi = O(t^{-\frac{1}{2}}) \quad \text{and} \quad \partial_x \Phi = O(t^{-1}\hat{\mathfrak{x}}_n), & (4.5a) \\ 0, \text{exactly}, \quad \text{if} \quad \Phi = (\text{real constant}) \times \mathfrak{F}_n, & (4.5b) \end{cases}$$

where, as before, $\hat{\mathfrak{x}}_n = \max(|\mathfrak{x}_n|, 1) = o(t^{\frac{1}{6}})$. An asterisk denotes the complex conjugate. It is noteworthy that, to lowest order, there is no oscillatory contribution involving $e^{\pm 2i\mathfrak{k}_n x}$. (This is not true of the remaining terms $O(t^{-1}\hat{\mathfrak{x}}_n)$.)

To obtain $\mathfrak{M}_{nn}$, we may set

$$\Phi = e^{-i\mathfrak{k}_n x}(-n^2 + \partial_x^2)(\mathfrak{F}_n e^{i\mathfrak{k}_n x}).$$

By (4.5b), the contributions $-n^2 \mathfrak{F}_n$ and $-\mathfrak{k}_n^2 \mathfrak{F}_n$ to this expression may be ignored, for the purpose of substituting into (4.5a). That is, we could equally well take

$$\begin{aligned} \Phi &= 2i\mathfrak{k}_n \partial_x \mathfrak{F}_n + \partial_x^2 \mathfrak{F}_n, \\ &= 2i\mathfrak{k}_n \partial_x \mathfrak{F}_n + O(t^{-1}\hat{\mathfrak{x}}_n) \end{aligned}$$

by (3.30c), which also implies that this Φ satisfies the conditions of (4.5a). Thence

$$\mathfrak{M}_{nn} = \mathfrak{e}_{nn}[\tfrac{1}{2}n\mathfrak{k}_n^2\,\partial_x|\mathfrak{F}_n^2| + O(t^{-1}\hat{\mathfrak{x}}_n)]. \tag{4.6a}$$

To calculate $\mathfrak{B}_{nn}$ we take

$$\Phi = \mathfrak{G}_n - K^2\mathfrak{F}_n = -i\mathfrak{k}_n\,\partial_x\mathfrak{F}_n + O(t^{-1}\hat{\mathfrak{x}}_n)$$

in (4.5a), noting (3.32), (4.5b) and that K is real. By (3.32), (3.30c) and (3.33), the conditions of (4.5a) are again satisfied and it follows that

$$\mathfrak{B}_{nn} = \mathfrak{e}_{nn}[-\tfrac{1}{4}n\mathfrak{k}_n^2\,\partial_x|\mathfrak{F}_n^2| + O(t^{-1}\hat{\mathfrak{x}}_n)]. \tag{4.6b}$$

An alternative and illuminating way of calculating $\mathfrak{M}_{nn}$ is to use the fact that it represents the curl of a body force which in turn equals the divergence of the Reynolds stress tensor. Without quoting details (but see the writer's (1972) paper) we note that the horizontal component of this body force turns out to be exactly independent of z, and therefore irrotational: $\mathfrak{M}_{nn}\sin(2nz)$ is entirely attributable to minus the x-derivative of the average, over a wavelength, of the contribution $-\partial_z\{(\partial_x\psi)^2\}$ to the vertical force. (But because of the equal importance of the non-mechanical forcing $\mathfrak{B}_{nn}$, the second-order motion cannot be described solely as a response to a 'radiation stress' as in the problems considered by Longuet-Higgins & Stewart (1964) – at least, not from an Eulerian viewpoint.)

4.3. *Solution*

We could now solve (4.1) by means of transform methods as in §3. However, it is simpler for our purposes to proceed as follows.

Consider the solution to the 'long-wave equations', namely (4.1) with ∇^2 replaced by ∂_z^2. Denoting the long-wave solution by

$$\overline{\Psi} = \sum_{n=1}^{n_K}\overline{\Psi}_{2n}\sin(2nz), \quad \overline{\Theta} = \sum_{n=1}^{n_K}\overline{\Theta}_{2n}\sin(2nz), \tag{4.7}$$

we can separate out the problem for each component as

$$\begin{cases}(\partial_t+\partial_x)(-4n^2\overline{\Psi}_{2n}) + \partial_x\overline{\Theta}_{2n} = \mathfrak{M}_{nn}(x,t), & (4.8a)\\ (\partial_t+\partial_x)\,\overline{\Theta}_{2n} + C_{2n}^2\partial_x(-4n^2\overline{\Psi}_{2n}) = \mathfrak{B}_{nn}(x,t), & (4.8b)\\ \overline{\Psi}_{2n} = \overline{\Theta}_{2n} = 0 \quad \text{for} \quad t<0, & (4.8c)\\ \partial_x\overline{\Psi}_{2n},\quad \overline{\Theta}_{2n}\to 0 \quad \text{as} \quad |x|\to\infty \quad (t<\infty). & (4.8d)\end{cases}$$

As can easily be verified, the solution to this nondispersive problem is

$$-4n^2\overline{\Psi}_{2n}(x,t) = \Lambda_{2n}(\mathfrak{M}_{nn}, C_{2n}^{-1}\mathfrak{B}_{nn}), \tag{4.9a}$$

$$\overline{\Theta}_{2n}(x,t) = \Lambda_{2n}(\mathfrak{B}_{nn}, C_{2n}\mathfrak{M}_{nn}), \tag{4.9b}$$

where the operator $\Lambda_{2n}(u,v)$ is defined, for any given pair of functions $u(x,t)$, $v(x,t)$ by

$$\Lambda_{2n}(u,v) = \Lambda_{2n}^+(u,v) + \Lambda_{2n}^-(u,v); \tag{4.10a}$$

$$\Lambda_{2n}^{\pm}(u,v) = \frac{1}{2}\int_0^t\{u(x+C_{2n}^{\pm}\tau - C_{2n}^{\pm}t,\ \tau) \pm v(x+C_{2n}^{\pm}\tau - C_{2n}^{\pm}t,\ \tau)\}\,d\tau. \tag{4.10b}$$

One can now use (4.9) to obtain to leading order the columnar-disturbance strengths in the $2n$th modal component, Ψ_{2n}, Θ_{2n} say, of the *exact* solution of (4.1). First, it can easily be shown that the expressions (4.9) do contain columnar disturbances with strengths of order unity as $t \to \infty$; explicit expressions are given below. Second, it can be shown that the differences

$$\Psi'_{2n} \equiv \Psi_{2n} - \overline{\Psi}_{2n}, \quad \overline{\Theta}'_{2n} = \Theta_{2n} - \overline{\Theta}_{2n}$$

between (4.9) and the exact solution contain relatively negligible, if any, columnar disturbances. A simple proof of this fact is given in appendix B.

To find explicitly the strengths of the columnar disturbances described by (4.9) we confine attention to regions of space such that

$$x = x_0 + Vt \quad (V \neq \mathfrak{v}_n;\ V \neq C_{2n}^{\pm}), \tag{4.11}$$

x_0 and V being constants. Now it follows from the properties (3.30a, b), and their consequence

$$\partial_t|\mathfrak{F}_n^2| = -\mathfrak{v}_n \partial_x|\mathfrak{F}_n^2| + O(t^{-1}\hat{\mathfrak{x}}_n) \quad [\hat{\mathfrak{x}}_n \equiv \max(|\mathfrak{x}_n|, 1)]$$

that, when x is given by (4.11) and t is sufficiently large (depending on x_0),

$$\int_0^t [\mathfrak{e}_{nn}(\xi,\tau)\{\partial_\xi|\mathfrak{F}_n^2(\xi,\tau)| + O(t^{-1}\hat{\mathfrak{x}}_n)\}|_{\xi = x + C_{2n}^{\pm}\tau - C_{2n}^{\pm}t}]\,d\tau$$
$$= -\mathfrak{b}_n^{\pm}\{|C_{2n}^{\pm} - \mathfrak{v}_n|^{-1}|\mathfrak{a}_n^2| + O(\mathfrak{X}^{-1}) + O(t^{-\frac{1}{2}}\mathfrak{X}^2)\},$$

where $\mathfrak{b}_n^{\pm}$ is equal to 1 if V lies between $\mathfrak{v}_n$ and $C_{2n}^{\pm}$, and is zero otherwise. (The first error term comes from that in (3.30b) at the ends of the significant part of the range of integration, and the second from integration of the error estimates in the integrand and in the relation below (4.11).) The total error estimate will be minimized at $O(t^{-\frac{1}{6}})$ if we choose

$$\mathfrak{X} = t^{\frac{1}{6}}. \tag{4.12}$$

One can now easily calculate the result of substituting (4.6) into (4.9) given (4.11) and (4.12). The result is that, for t sufficiently large

$$\left.\begin{aligned} \overline{\Psi}_{2n}(x_0 + Vt, t) &= \overline{\Psi}_{2n}^{+}(V) + \overline{\Psi}_{2n}^{-}(V) \\ \overline{\Theta}_{2n}(x_0 + Vt, t) &= \overline{\Theta}_{2n}^{+}(V) + \overline{\Theta}_{2n}^{-}(V) \end{aligned}\right\}; \tag{4.13a}$$

where
$$\overline{\Psi}_{2n}^{\pm} = \overline{\Theta}_{2n}^{\pm} = 0 \quad \text{when} \quad V \notin (\mathfrak{v}_n, C_{2n}^{\pm}), \tag{4.13b}$$

$$\left.\begin{aligned} -4n^2\overline{\Psi}_{2n}^{\pm} &= \tfrac{1}{2}|C_{2n}^{\pm} - \mathfrak{v}_n|^{-1}(M_{nn} \pm C_{2n}^{-1}B_{nn}) \\ \overline{\Theta}_{2n}^{\pm} &= \tfrac{1}{2}|C_{2n}^{\pm} - \mathfrak{v}_n|^{-1}(B_{nn} \pm C_{2n}M_{nn}) \end{aligned}\right\} \quad \text{when} \quad V \in (\mathfrak{v}_n, C_{2n}^{\pm}), \tag{4.13c}$$

and
$$M_{nn} = -\tfrac{1}{2}n\mathfrak{k}_n^2|\mathfrak{a}_n|^2 + O(t^{-\frac{1}{6}}), \tag{4.13d}$$

$$B_{nn} = +\tfrac{1}{4}n\mathfrak{k}_n^2|\mathfrak{a}_n|^2 + O(t^{-\frac{1}{6}}). \tag{4.13e}$$

We note from (4.4b) and (3.18) that $C_{2n}^{+} \neq \mathfrak{v}_n$; but C_{2n}^{-} can be equal to $\mathfrak{v}_n$ if, exceptionally,

$$K = 2^{\frac{1}{2}}n \quad (\text{some integer } n). \tag{4.14}$$

(4.13b) then suggests that, for this n, the disturbance corresponding to the lower signs is confined to the nth tail. Returning to the more general expression (4.9), we can confirm this and also show that the amplitude of the disturbance grows as $t^{\frac{1}{2}}$. (Even when such resonant growth occurs it will not invalidate the basic amplitude expansion, under the restriction (7.1) which must in any case be imposed, as discussed in § 7.)

4.4. *Physical interpretation of the columnar-disturbance formulae*

In general (when (4.14) does not hold) the simplicity of the formulae (4.13) can be accounted for by the following considerations, which also provide an independent check on the foregoing analysis.

The four expressions for $\overline{\Psi}^{\pm}_{2n}$ and $\overline{\Theta}^{\pm}_{2n}$ given by (4.13*c*) are equivalent to the four relations

$$\overline{\Theta}^{-}_{2n} = 4n^2C_{2n}\overline{\Psi}^{-}_{2n}, \quad \overline{\Theta}^{+}_{2n} = -4n^2C_{2n}\overline{\Psi}^{+}_{2n}, \tag{4.15a, b}$$

$$|C^{-}_{2n}-\mathfrak{v}_n|(-4n^2\overline{\Psi}^{-}_{2n})+|C^{+}_{2n}-\mathfrak{v}_n|(-4n^2\overline{\Psi}^{+}_{2n}) = M_{nn}, \tag{4.15c}$$

$$|C^{-}_{2n}-\mathfrak{v}_n|\overline{\Theta}^{-}_{2n}+|C^{+}_{2n}-\mathfrak{v}_n|\overline{\Theta}^{+}_{2n} = B_{nn}. \tag{4.15d}$$

The first pair of relations expresses the fact that the far end of each columnar disturbance propagates freely, with intrinsic velocity $-C_{2n}$ or $+C_{2n}$, as described by the homogeneous counterparts of the long-wave equations (4.8*a*, *b*). Now

$$M_{nn} = \int_{-\infty}^{\infty}\mathfrak{M}_{nn}\,dx+O(t^{-\frac{1}{3}}) \quad \text{and} \quad B_{nn} = \int_{-\infty}^{\infty}\mathfrak{B}_{nn}\,dx+O(t^{-\frac{1}{3}});$$

consequently the second pair of relations (4.15*c*, *d*) exhibits the way in which the solution, at large time, approximately satisfies the following integral relations, which are an immediate consequence of either the long-wave or the full equations under appropriate boundary conditions:

$$\partial_t\int_{-\infty}^{\infty}\{-4n^2\Psi_{2n},\Theta_{2n}\}\,dx = \int_{-\infty}^{\infty}\{\mathfrak{M}_{nn},\mathfrak{B}_{nn}\}\,dx. \tag{4.16a, b}$$

The right-hand side of (4.16*a*), $\int\mathfrak{M}_{nn}\,dx$, is the total rate at which vorticity with vertical distribution $\sin(2nz)$ is being introduced into the fluid; this takes place within the comparatively small region $|\mathfrak{x}_n| \leqslant t^{\frac{1}{3}}$, i.e. $|x-\mathfrak{v}_nt| \leqslant t^{\frac{2}{3}}$. Relation (4.15*c*) shows that at large time most of this vorticity is being shared between, and distributed evenly within, the two expanding regions occupied by the two columnar disturbances. Similarly, $\int\mathfrak{B}_{nn}\,dx$ is the total rate of introduction of buoyancy into the $\sin(2nz)$ mode, and (4.15*d*) describes how this is taken up predominantly by the two columnar disturbances. The fact that these integrals are different from zero is the reason why the problem (2.10) is of type one.

The fact that the first-order problem (2.9) is of type zero can be understood similarly, after transferring the forcing term from the boundary condition (2.9*d*) to the equations by subtracting from $\psi^{(1)}$ a function satisfying the boundary conditions and linear in z. Then we obtain on the right-hand sides of (2.9*a*) and (2.9*b*) forcing terms proportional to $(\partial_t+\partial_x)^2\,\partial_x\,\mathsf{h}$ and $(\partial_t+\partial_x)\,\mathsf{h}$ respectively. The integral of the former with respect to x is zero, and that of the latter tends to zero as $\mathsf{h}(x,t)$ becomes steady.

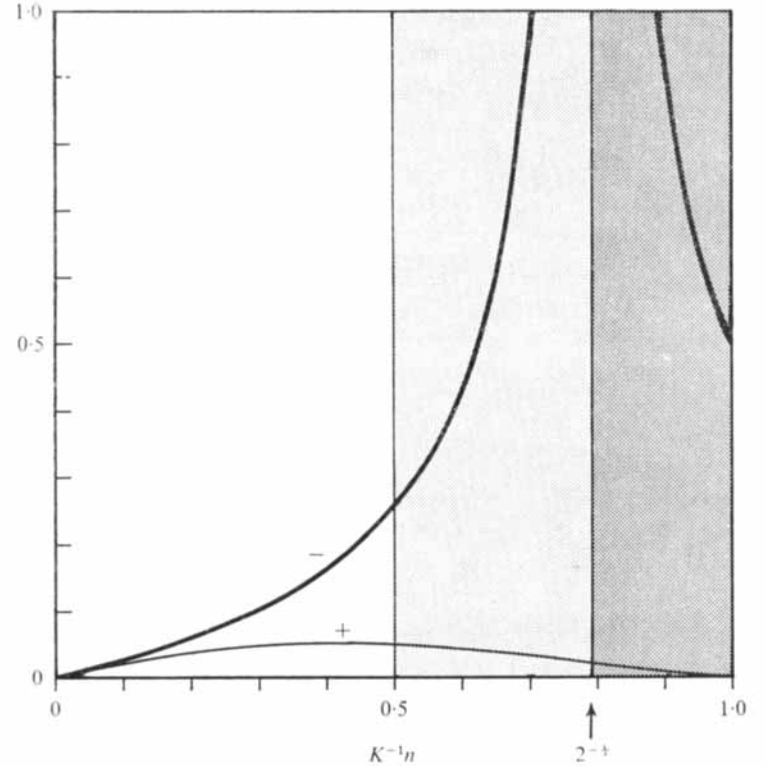

FIGURE 4. Strengths of the pair of columnar disturbances due to the nth lee-wave tail, as a function of $K^{-1}n$. Upon multiplying the ordinate by $\mathfrak{k}_n^2|\mathfrak{a}_n|^2$, the heavy curve gives Γ_{2n}^-, and the light curve gives Γ_{2n}^+. (See text.)

4.5. *Summary of main results*

We now gather in a form convenient for reference the principal results of the foregoing analysis. These results do not depend on the precise shapes $\mathfrak{F}_n(x,t)$ of the lee-wave tails, but only on the properties (3.30)–(3.33) of $\mathfrak{F}_n$ and $\mathfrak{G}_n$. In what follows it will be useful to recall that $\mathfrak{v}_n(>0)$, $C_{2n}(>0)$ and $C_{2n}^{\pm}$ are defined by (3.18), (4.4a) and (4.4b) respectively, and that n_K is the largest integer less than K.

Each of the n_K lee-wave tails moves downstream with speed $\mathfrak{v}_n$ and forces a pair of second-order disturbances. The free end of one of these propagates with the stream, with velocity C_{2n}^+, and that of the other against it, with velocity C_{2n}^-. The disturbance propagating against the stream is described asymptotically, for $x \sim Vt\ (t \to \infty,\ V \neq \mathfrak{v}_n,\ V \neq C_{2n}^-)$, by

$$\text{stream function} \sim \epsilon^2(2n)^{-1}\,\Gamma_{2n}^-\sin(2nz) \quad [V \in (\mathfrak{v}_n, C_{2n}^-)], \tag{4.17a}$$

$$\text{buoyancy} \sim \epsilon^2(2n)\,C_{2n}\,\Gamma_{2n}^-\sin(2nz) \quad [V \in (\mathfrak{v}_n, C_{2n}^-)], \tag{4.17b}$$

both being zero for $V \notin (\mathfrak{v}_n, C_{2n}^-)$. Here (taking the lower signs in (4.13c))

$$\Gamma_{2n}^- \equiv \frac{\mathfrak{k}_n^2|\mathfrak{a}_n^2|\{2C_{2n}+1\}}{16C_{2n}|C_{2n}^- - \mathfrak{v}_n|} = \frac{\mathfrak{k}_n^2|\mathfrak{a}_n^2|K^{-1}n\{1+K^{-1}n\}}{4|1-2(K^{-1}n)^3|}, \tag{4.17c}$$

where $\epsilon\mathfrak{a}_n$ is the steady amplitude of the stream function $\epsilon\,\mathrm{Re}\,(\mathfrak{a}_n e^{i\mathfrak{k}_n x})$ for the

nth lee-wave train. The dependence of Γ_{2n}^{-} upon $K^{-1}n$ is shown as the upper curve in figure 4.

The expressions (4.17) represent a columnar disturbance that penetrates upstream past the obstacle if

$$C_{2n}^{-} \equiv 1-(2n)^{-1}K < 0, \tag{4.18}$$

corresponding to the unshaded part of figure 4. It should be noted that this can occur only in a multiply-subcritical case

$$K > 2.$$

The corresponding alteration to the upstream velocity profile is

$$\epsilon^2 \Sigma\, \Gamma_{2n}^{-} \cos(2nz),$$

where the summation is over those values of n for which (4.18) is true. It is noteworthy that the sign of the velocity-profile alteration is always such as to increase the total velocity near the boundaries (but it should be mentioned that this does not necessarily apply to the rotating-tube problem of appendix A). Such a change is opposite to what would be expected from a naive argument about the obstacle tending to block the flow (which, of course, is *not* an apt description of the process described by the present theory, either in the stratified or the rotating problem).

For points within the shaded part of figure 4, the disturbance represented by (4.17) is confined downstream of the obstacle. The darker shading corresponds to a columnar disturbance extending downstream of the tail, and the lighter shading to one extending upstream of the tail but not reaching the obstacle. The dividing line between these two cases represents the exceptional case in which (4.14) holds, and in which the disturbance is not columnar, being confined to the tail that generates it and growing in amplitude as $t^{\frac{1}{2}}$.

The other member of the nth pair of second-order disturbances propagates with the stream, and so always takes the form of a columnar disturbance extending downstream of the tail that generates it, since $\mathfrak{v}_n < C_{2n}^{+}$ always. It is described, for $x \sim Vt$ ($t \to \infty$, $V \neq \mathfrak{v}_n$, $V \neq C_{2n}^{+}$), by

$$\text{stream function} \sim \epsilon^2(2n)^{-1}\Gamma_{2n}^{+}\sin(2nz) \quad [V \in (\mathfrak{v}_n, C_{2n}^{+})], \tag{4.19a}$$

$$\text{buoyancy} \sim -\epsilon^2(2n)\,C_{2n}\,\Gamma_{2n}^{+}\sin(2nz) \quad [V \in (\mathfrak{v}_n, C_{2n}^{+})], \tag{4.19b}$$

both being zero for $V \notin (\mathfrak{v}_n, C_{2n}^{+})$, where

$$\Gamma_{2n}^{+} \equiv \frac{\mathfrak{k}_n^2|\mathfrak{a}_n^2|\{2C_{2n}-1\}}{16C_{2n}|C_{2n}^{+}-\mathfrak{v}_n|} \equiv \frac{\mathfrak{k}_n^2|\mathfrak{a}_n^2|K^{-1}n\{1-K^{-1}n\}}{4|1+2(K^{-1}n)^3|}, \tag{4.19c}$$

shown as the lower curve in figure 4.

In the numerators of (4.17c) and (4.19c) the first term within curly brackets arises from the mechanical forcing $\mathfrak{M}_{nn}$ and the second term from the buoyancy forcing $\mathfrak{B}_{nn}$. As one might guess, it is only the first of these that is associated with the way in which the fluid satisfies the *impulse principle* (Benjamin 1970, eq. (3.13)). It is found that the $\mathfrak{B}_{nn}$ contributions cancel when (4.17a) and (4.19a) are used to calculate the total impulse to leading order. (The latter's time rate of change, as can readily be checked, does of course turn out to be minus the drag given by (3.43).)

To obtain an idea of the numerical strength of the upstream influence, take $K = 2{\cdot}5$, for example; we then find that the upstream velocity profile is changed by $0{\cdot}16\mathfrak{k}_1^2|\mathfrak{a}_1|^2\cos(2z)$. In dimensional terms the value of this at, say, the boundaries, becomes $0.16U$ times the square of the maximum streamline or isotherm slope associated with the $(n = 1)$ lee-wave mode.

4.6. *Independence of initial development*; *slow introduction of obstacle*

We state two extensions of the foregoing, which provide evidence that our main results concerning upstream influence are largely independent of the way in which the obstacle is introduced into the initially undisturbed flow.

(i) The main results given above are clearly unaltered if the time-dependence of the obstacle $\mathsf{h}(x,t)$ has a Laplace transform with a simple pole at $p = 0$ and, at all its other singularities, $\mathrm{Re}\,p < -b$ where b is a positive number. This merely introduces into (3.7) and (3.11) additional contributions that are $O(e^{-bt})$ for large time. Consequently the same asymptotic columnar-disturbance strengths are obtained for a variety of ways of making $\mathsf{h}(x,t)$ tend (exponentially) to $h(x)$ as $t \to \infty$.

(ii) The case $\mathsf{h}(x,t) = h(x)$ times a slowly varying function of time,

$$S(t) \quad [S(-\infty) = 0]$$

is not included in (i), but is amenable to the usual formal two-scale approach. Each lee-wave train is given by the steady solution modulated by an envelope $S(t-\mathfrak{v}_n^{-1}x)$. The 'tail' now comprises the whole lee-wave train. The resulting second-order disturbance, to leading order, is

$$\begin{aligned}\epsilon^2\psi^{(2)} = \sum_{n=1}^{n_K} \frac{\epsilon^2(2n)^{-1}\,\Gamma_{2n}^-\sin(2nz)}{\mathrm{sgn}\,(C_{2n}^- - \mathfrak{v}_n)}\left[\mathrm{sgn}\,(C_{2n}^-)\,H\!\left(\frac{x}{C_{2n}^-}\right)S^2\!\left(t-\frac{x}{C_{2n}^-}\right) - H(x)\,S^2\!\left(t-\frac{x}{\mathfrak{v}_n}\right)\right] \\ + \sum_{n=1}^{n_K}\epsilon^2(2n)^{-1}\,\Gamma_{2n}^+\sin(2nz)\,H(x)\left[S^2\!\left(t-\frac{x}{C_{2n}^+}\right) - S^2\!\left(t-\frac{x}{\mathfrak{v}_n}\right)\right] \quad (C_{2n}^+ > \mathfrak{v}_n > 0)\end{aligned} \tag{4.20}$$

where $H(x)$ is Heaviside's step function; $\epsilon^2\theta^{(2)}$ is given by the same expression modified by replacing $(2n)^{-1}$ by $2n\,C_{2n}$ and inserting a minus sign before the second line. For those disturbances that penetrate upstream this solution evidently reduces to (4.17) with Γ_{2n}^- replaced by its natural generalization

$$\Gamma_{2n}^- \times S^2\!\left(t-\frac{x}{C_{2n}^-}\right) \quad (x < 0, C_{2n}^- < 0), \tag{4.21}$$

V now being irrelevant.

5. The remaining contributions to $\psi^{(2)}$ and $\theta^{(2)}$

Denote these by $\psi(x,z,t)$ and $\theta(x,z,t)$. That is,

$$\psi^{(2)} = \Psi + \psi; \quad \theta^{(2)} = \Theta + \theta,$$

where Ψ and Θ are the functions investigated in §4. Then ψ, θ is the solution of a difference problem

$$\begin{cases} (\partial_t + \partial_x)\nabla^2\psi + \partial_x\theta = \mathsf{M}(x,z,t), & (5.1a) \\ (\partial_t + \partial_x)\theta - K^2\partial_x\psi = \mathsf{B}(x,z,t), & (5.1b) \\ \psi = \theta = 0 \quad \text{for} \quad t < 0, & (5.1c) \\ \partial_x\psi(x,0,t) = \partial_x h(x,t), & (5.1d) \\ \psi(x,\pi,t) = 0, & (5.1e) \\ \partial_x\psi,\ \theta \to 0 \quad \text{as} \quad |x| \to \infty \quad (t < \infty), & (5.1f) \end{cases}$$

where
$$\mathsf{M} \equiv \mathscr{M} - \sum_{n=1}^{n_K} \mathfrak{M}_{nn} \sin(2nz), \tag{5.2a}$$

$$\mathsf{B} \equiv \mathscr{B} - \sum_{n=1}^{n_K} \mathfrak{B}_{nn} \sin(2nz). \tag{5.2b}$$

$\mathfrak{M}_{nn}$ and $\mathfrak{B}_{nn}$ are defined in (4.3); $\mathscr{M}$ and $\mathscr{B}$ are defined by (2.12), in terms of the first-order solutions, which are

$$\psi^{(1)} = \psi_{s(1)} + \psi_{s(\infty)} + \sum_{n=1}^{\infty} (\psi_n^- + \psi_n^+) \sin(nz) \tag{5.2c}$$

and a similar expression for $\theta^{(1)}$, where the various terms are defined in (3.40), (3.42), and (3.34).

That the columnar-disturbance strengths (4.17) and (4.19) should dominate those contained in ψ and θ is intuitively plausible from the preliminary discussion in §1. It is far from obvious mathematically, if only because of the large spatial extent (order t) of the region over which the forcing functions M and B are significant. We have been able to prove, however, using techniques that avoid detailed solution of (5.1), that the strengths of columnar disturbances present in ψ and θ are

$$O(t^{-\frac{1}{3}} \log t). \tag{5.3}$$

This estimate is marginally greater than the error terms $O(t^{-\frac{1}{3}})$ already present in (4.17) and (4.19) (see (4.13)), and is therefore the governing estimate.

It is conjectured that (5.3) is far from the sharpest possible estimate. By going into more detail or by using deeper (but perhaps less physically intelligible) methods, the estimate (5.3) could probably be improved to

$$O(t^{-1}) \tag{5.4}$$

(which, it can be argued heuristically, is the actual strength to be expected for columnar disturbances due to self-interaction of the transients in regions far removed from the lee-wave tails).

The method used to establish (5.3) will now be indicated in brief outline. The forms of $\psi^{(1)}$ and $\theta^{(1)}$, and the Jacobian form of the nonlinear terms, are such that we can write

$$\mathsf{M} = \sum_{n=1}^{\infty} \mathsf{M}_{nn}(x,t)\sin(2nz) + \sum_{m\neq n} \mathsf{M}_{mn+}(x,t)\sin\{(m+n)z\} + \sum_{m\neq n} \mathsf{M}_{mn-}(x,t)\sin\{(m-n)z\} \quad (5.5)$$

and similarly for B. In the last two summations, both m and n run from 1 to ∞. In virtue of (B. 6), it is enough to estimate columnar-disturbance strengths in

$$\left.\begin{aligned} \overline{\Psi}_{2n} &\equiv -\tfrac{1}{4}n^{-2}\Lambda_{2n}(\mathsf{M}_{nn}, C_{2n}^{-1}\mathsf{B}_{nn}), \\ \overline{\Psi}_{mn\pm} &\equiv -(m\pm n)^{-2}\Lambda_{m\pm n}(\mathsf{M}_{mn\pm}, C_{m\pm n}^{-1}\mathsf{B}_{mn\pm}) \end{aligned}\right\} \quad (5.6)$$

and similar expressions $\overline{\theta}_{2n}$ and $\overline{\theta}_{mn\pm}$, provided that

$$\int_0^t \left[\int_{-\infty}^{\infty} \{(\partial_t + \partial_x)\overline{\Psi}_{2n}\}^2\,dx\right]^{\frac{1}{2}} dt = O(t^{\frac{7}{8}}\log t), \quad (5.7)$$

and similarly for $\overline{\Psi}_{mn\pm}$, $\overline{\theta}_{2n}$, and $\overline{\theta}_{mn\pm}$. That (5.7) is certainly satisfied can be shown to follow from straightforward estimates based on the results of §3. Given (5.7), relation (B 6) then implies that the strengths of any columnar disturbances *not* described by (5.6) are $O(t^{-\frac{1}{8}}\log t)$.

The analysis of (5.6) is based on the principle that if

$$\int_{x_1(t)}^{x_2(t)} \overline{\Psi}_{2n}\,dx = O(\iota t) \quad \text{for all} \quad x_1 = O(t),\ x_2 = O(t), \quad (5.8)$$

then any columnar disturbances present in $\overline{\Psi}_{2n}$ must have strength $O(\iota)$. Upon substituting from (5.6) and (4.10) it is seen that (5.8) and its analogue for $\overline{\theta}_{2n}$ will certainly be satisfied if relations of the form

$$t^{-1}\iint_{\mathscr{D}(t)} \mathsf{M}_{nn}(x,t)\,dt\,dx = O(\iota), \quad t^{-1}\iint_{\mathscr{D}(t)} \mathsf{B}_{nn}(x,t)\,dt\,dx = O(\iota), \quad (5.9a,b)$$

hold for all parallelograms $\mathscr{D}(t)$ of the type illustrated by the broken lines in figure 3 above. One pair of $\mathscr{D}$'s sides has slope $C_{2n}^{\pm}$, and $\mathscr{D}$ can expand with time subject to its perimeter and area being $O(t)$ and $O(t^2)$ respectively. Similar relations involving $\mathsf{M}_{mn\pm}$ and $\mathsf{B}_{mn\pm}$ will bound the columnar-disturbance strengths in $\overline{\Psi}_{mn\pm}$ and $\overline{\theta}_{mn\pm}$.

All these integrals are estimated from (3.20) and (3.30)–(3.33), together with similar estimates for the purely-transient contributions, derived from expressions like (3.37). The result is (5.3). It is dominated by the contributions to M_{nn} ($n \leqslant n_K$) from the interaction between transients and steady lee waves for $\mathfrak{x}_n$ just less than $-\mathfrak{X} = -t^{\frac{1}{8}}$.

The largeness of the domain of integration $\mathscr{D}(t)$ means that a certain amount of delicacy is needed in making the estimates. To obtain (5.3) one must exploit

(i) the vanishing of the self-interaction of the steady part $\psi'_s(x,z)$, $\theta'_s(x,z)$ of the first-order solution defined by (3.44): it can easily be verified that

$$\nabla^2\psi'_s = -K^2\psi'_s = -\theta'_s; \tag{5.10a}$$

K is real, so that, as in (4.5*b*)

$$\frac{\partial(\psi'_s, \nabla^2\psi'_s)}{\partial(x,z)} = \frac{\partial(\psi'_s, \theta'_s)}{\partial(x,z)} = 0 \tag{5.10b}$$

(the basis of Long's model),

(ii) a similar cancellation among the *leading terms* of the transient self-interactions,

(iii) a near-cancellation among the leading terms of each interaction between transients with $n \leqslant n_K$, and steady lee-waves (e.g. the two terms of (3.34)), at locations near the downstream (tail) end of region IV in the version of figure 3 appropriate to the transients (which will be different from that appropriate to the steady lee waves if the mode numbers n are respectively different),

(iv) the fact that the interactions just mentioned, and also those between transients with different values of n, make contributions to the integrands that oscillate in space-time, allowing application of a generalized, two-dimensional version of Riemann's Lemma (Jeffreys & Jeffreys 1962, §14.03).

Further details are not reproduced here, since they are lengthy. The main steps are given in the longer version of §5 mentioned in the footnote to §3.3, which is available from the *J.F.M.* editorial office.

The remaining contribution to ψ and θ is that from the boundary forcing $\hbar$ on the right of (5.1*d*). From (2.11),

$$\hbar(x,t) = -\mathsf{h}(x,t)\,\partial_z\psi^{(1)}(x,0,t).$$

It is fairly obvious that no columnar disturbances arise from this forcing, because of the evanescence of h and therefore of $\hbar$ as $|x| \to \infty$. The problem for this contribution is entirely similar to the first-order problem solved in §3.

Alternatively, the absence of significant columnar disturbances due to $\hbar$ can be verified using (4.10) (see the argument sketched at the end of §4.4), together with the result of appendix B. These give $O(t^{-\frac{1}{2}})$, a sufficient, if crude, estimate.

6. Viscous and unbounded lee-wave régimes

We now briefly discuss the inviscid, unbounded case, and the effect of slight viscosity in the bounded case when there is no boundary-layer separation. This complements an analysis by Miles (unpublished, but similar to that of Miles 1970) which indicates validity of LH for unbounded, steady, slightly-viscous flow. The transient, inviscid, unbounded problem is similar to the problem for two-dimensional wave packets considered by Bretherton (1969, §3.3).

Note that a viscous formulation of LH cannot be expected to be equivalent to a transient formulation, if only because the problem is nonlinear. In the bounded case the answers (there are *two* viscosity-influenced asymptotic régimes) are indeed quite different from the result of §4.

6.1. *The two slightly-viscous bounded régimes*

Suppose now that the fluid is slightly viscous, with

$$E \equiv \nu/ND^2 \ll 1, \tag{6.1}$$

and with a buoyancy diffusivity of magnitude similar to that of the viscosity ν. The boundaries are taken as no-slip and moving with the basic flow, as in the usual 'towing' experiment. We assume no separation and thus no form drag apart from the (order-ϵ^2) wave drag.

According to linearized theory, the lee-wave system is steady and spatially attenuated for $t \gg E^{-\frac{1}{2}}$ because of the Stokes layers at the boundaries (Phillips 1966). (These layers are of thickness $E^{\frac{1}{2}}$, because particle oscillation frequencies are of order N.) For the problem of columnar-disturbance generation two asymptotic régimes are relevant, namely $E^{-\frac{1}{2}} \ll t \ll E^{-1}$ and $t \gg E^{-1}$. In the first régime any columnar disturbances that may exist are effectively inviscid; in the second they are fully viscous with a new and more complicated z-structure depending on the buoyancy diffusivity as well as on ν. A detailed analysis has not been carried out; the mathematics can be expected to be similar to that in Pedley (1969).

However, it can be seen without considering details that, in either régime, the excitation of columnar disturbances is only $O(\epsilon^2)$ *times a positive power of E*. This is because the lee-wave system is steady. Outside the Stokes layers (5.10) is relevant, except that there is an error $O(E)$ in (5.10a) so that (5.10b) is replaced by

$$\mathscr{M} = O(E) \quad \text{and} \quad \mathscr{B} = O(E), \quad \text{everywhere}, \tag{6.2}$$

reflecting the fact that if it were not for the viscous and diffusive terms in the interior, the nonlinear terms would vanish exactly, as in Long's model. (Of course one can verify that the result of explicitly performing the Stokes-layer analysis agrees with (6.2). What happens is that the interior velocity and buoyancy fields induced by the Stokes layers give rise to a contribution, of order $\epsilon^2 E^{\frac{1}{2}}$, to the nonlinear terms, which cancels that due to the spatial variation of amplitude, given by (4.6).)

The length of the steady lee-wave system is of order $E^{-\frac{1}{2}}$, and so the columnar-disturbance strength due to $\mathscr{M}$ and $\mathscr{B}$ is

$$\epsilon^2 \times E^{-\frac{1}{2}} \times O(E), \quad = O(\epsilon^2 E^{\frac{1}{2}}),$$

which vanishes in the inviscid limit.

There could also be columnar-disturbance generation, for $t \gg E^{-1}$, associated with (unseparated) obstacle wakes, or with the order-ϵ^2 rectified streaming in the Stokes boundary layers (Phillips 1966), but the strength of such columnar disturbances can still be expected to vanish with E. We estimate strengths $O(E^{\frac{1}{2}})$ and $O(\epsilon^2 E^{\frac{1}{4}})$ respectively.

6.2. *The inviscid, unbounded, transient régime*

Only the inviscid, time-dependent problem is mentioned here in view of Miles' result quoted above that a steady, slightly-viscous, unbounded analysis predicts vanishing upstream influence as $\nu \to 0$.

The importance of the *far* field in the inviscid bounded problem immediately suggests that, in unbounded geometry, any upstream influence will tend to zero as $t \to \infty$ because the lee-wave system, and hence the second-order forcing, now spreads out vertically as well as horizontally. Enough analysis has been carried out to confirm this and to show that the far-upstream horizontal disturbance velocity u is of order

$$\epsilon^2 t^{-2}. \tag{6.3}$$

Detailed solutions have not been obtained. The main features that differentiate the unbounded from the bounded problem are as follows, in brief outline.

If a finite obstacle is introduced into the inviscid, unbounded, uniformly-stratified flow of Long's model (Miles & Huppert 1969, and refs.; Pao 1969), linearized theory predicts a developing lee-wave system which at large times occupies a semicircular region with $0 \leqslant x \leqslant t$ as diameter. The contributions to $\mathscr{M}$ and $\mathscr{B}$ that are directly related to the lee-wave amplitude and so analogous to $\mathfrak{M}_{nn}$ and $\mathfrak{B}_{nn}$ are found to be significant only at the fringe of the semicircle. They force a disturbance whose height scale is set by the height $\frac{1}{2}t$ of the semicircle.

On this scale the buoyancy forcing $\mathscr{B}$ is found to have negligible effect in comparison with the mechanical forcing $\mathscr{M}$, in contrast with the bounded problem but in agreement with a tentative conclusion of Bretherton (1969, p. 796) in connection with his solution of a simpler but related problem. The group velocity of the second-order disturbance is nearly horizontal, and large, of order t. Therefore this disturbance very rapidly penetrates both far upstream and downstream.

As in Bretherton's problem, the large horizontal group velocity means first that the hydrostatic approximation applies, and second that the problem is approximately equivalent to one in which the forcing is concentrated at the z-axis so that

$$\mathscr{M} = t^{-2} g(\zeta)\,\delta(x) \quad (\zeta \equiv z/t)$$

(and $\mathscr{B}$ is irrelevant). Here $g(\zeta) = 0$ except in $0 \leqslant \zeta < \frac{1}{2}$; there it is an order-unity function of ζ apart from a behaviour like $(\frac{1}{2} - \zeta)^{-\frac{3}{2}}$ for ζ just less than $\frac{1}{2}$. At a fixed point, no matter how far upstream, order-of-magnitude considerations now show that the disturbance amplitude for u or θ is of order $\epsilon^2 t^{-2}$.

7. Restrictions on validity of the basic expansion and on meaning to be attached to Long's Hypothesis

Benjamin's impulse argument, our analysis, and LH as often conceived, all tacitly assume validity of the expansion (2.8) in powers of the small parameter ϵ. In particular they all depend upon the basic assumption that the time development (or steady viscous state) of the lee-wave system is uniformly approximated by the linearized description, at least qualitatively—so that, for instance, the finite-amplitude flow becomes steady and there is a constant wave drag.

Such an assumption is, however, likely to be justifiable only for dimensionless times satisfying

$$t = o(\epsilon^{-1}) \quad \text{as} \quad \epsilon \to 0 \tag{7.1}$$

(this being replaced in § 6.1 by an analogous condition restricting the smallness of ν). The existence of such restrictions seems clear from recent work on resonant interactions among internal gravity waves, noting the fact that stray transients (§ 3.3) will in general be present.

It is not resonant interactions between pairs of lee-wave trains that will initiate the breakdown after (7.1) is violated; their interaction coefficients are zero for basic flows admitting Long-type solutions, by (5.10). What is relevant is the unstable type of resonant interaction (e.g. McEwan 1971; Martin, Simmons & Wunsch 1972) between one lee-wave train and any of a large number of pairs of initially small transients. These interactions clearly cannot be described by the expansion procedure of § 2.2. They are negligible under (7.1), but for larger times will give rise to a cascade of lee-wave energy into more and more modes, leading to a complicated, and presumably unsteady, inviscid picture for $t \to \infty$ with ϵ constant.

8. Concluding remarks

The main results for the two-dimensional inviscid, transient, bounded problem have been summarized on pages 229–231. The corresponding results for the rotating-tube problem are set out in appendix A; in the rotating-tube problem, there is upstream influence even when only one lee-wave mode is present, since the combinations of Bessel functions involved have no property analogous to the orthogonality of $\sin(2z)$ and $\sin z$.

We remark that the result concerning the absence of upstream influence in the singly-subcritical two-dimensional problem appears to be true not only of $\psi^{(2)}$ but also of $\psi^{(3)}, \psi^{(4)}, \ldots$, i.e. formally true to all orders ϵ^r. An argument by induction on r, not reproduced here, indicates that every columnar disturbance contained in $\psi^{(r)}$ must have even-order modal structure $\sin(2nz)$ and must consequently be confined downstream, when $n_K = 1$. But the restriction discussed in § 7 should be borne in mind.

The results of § 4 and appendix A describe the way in which the impulse principle is satisfied by the fluid motion. We note that the total impulse is equal to the formal 'wave momentum' for the lee waves, defined as wave energy divided by horizontal phase velocity (R. W. Stewart, cited in Bretherton 1969, §4.1). Our results, however, imply a rather unexpected relationship between this formal wave momentum and the actual intrinsic properties of the waves, which is discussed elsewhere (McIntyre 1972).

In laboratory experiments, columnar disturbances could easily result from other causes, such as drag associated with separation bubbles, or local turbulent redistribution of x-momentum or of buoyancy or angular momentum. Any of these could correspond to the presence of forcing terms of type one (see § 4.4). In order to observe the effect discussed in the present paper, a carefully designed laminar experiment with a streamlined obstacle would appear to be necessary. It can be seen from figure 4 that one of the stronger and more easily-observable columnar disturbances is predicted to occur just downstream of the lee waves, when C_2^- is just greater than $\mathfrak{v}_1$, i.e. when

$$K(\equiv ND/U) \quad \text{is just less than } 2^{\frac{1}{3}}, = 1{\cdot}26, \tag{8.1}$$

D being π^{-1} times the depth of the fluid. The analogue of this in the rotating-tube problem is given by (A 2) and (A 18).

The writer acknowledges helpful conversations or correspondence with T. B. Benjamin, P. G. Drazin, H. E. Huppert, J. J. Mahony, J. W. Miles, K. Stewartson, and K. Trustrum. He is indebted to A. W. Stewart for calculating table 1, using the facilities of the Cambridge University Computer Laboratory, and would like to thank St John's College, Cambridge, for support in the form of a research fellowship.

Appendix A. Results for the rotating-tube problem

In place of the Cartesian co-ordinates (x, z), define axial and radial cylindrical polar co-ordinates (x, r). Making appropriate changes in the definitions of various symbols we have, in place of (2.3),

$$(\partial_t + \partial_x)\,\eta + \partial_x\chi = \frac{\partial(r^{-1}\psi, \eta)}{\partial(x, r)} - \partial_x\left(\frac{\eta\psi}{r^2} + \frac{\chi^2}{K^2 r}\right), \tag{A 1a}$$

$$(\partial_t + \partial_x)\,\chi - \frac{K^2}{r}\,\partial_x\psi = \frac{1}{r^2}\frac{\partial(\psi, r\chi)}{\partial(x, r)}. \tag{A 1b}$$

Here

$$K \equiv NR/U. \tag{A 2}$$

N is twice the basic rotation rate about the x-axis, and R is the tube radius, used as the length scale for nondimensionalization. The azimuthal component of vorticity

$$\eta \equiv r^{-1}\partial_x^2\psi + \partial_r(r^{-1}\,\partial_r\psi); \tag{A 3}$$

the stream function is now defined by

$$\left.\begin{aligned} x \text{ component of velocity} &= 1 + r^{-1}\,\partial_r\psi, \\ r \text{ component of velocity} &= -r^{-1}\,\partial_x\psi, \end{aligned}\right\} \tag{A 4}$$

where the velocity scale is U; χ is the relative azimuthal velocity measured in the same sense as the basic rotation, and made dimensionless with the scale $K^{-1}U$.

Let the complex amplitude of the nth lee-wave train be denoted by $\mathfrak{a}_n$ as before. The tail is described by

$$\psi = \epsilon r J_1(j_n r)\,\mathrm{Re}\,\{\mathfrak{F}_n(x, t)\,e^{i\mathfrak{k}_n x}\}, \tag{A 5a}$$

$$\chi = \epsilon J_1(j_n r)\,\mathrm{Re}\,\{\mathfrak{G}_n(x, t)\,e^{i\mathfrak{k}_n x}\}, \tag{A 5b}$$

where

$$j_1 = 3{\cdot}83, \quad j_2 = 7{\cdot}02, \ldots, j_n \sim (n + \tfrac{1}{4})\,\pi, \tag{A 6}$$

the zeros of the Bessel function J_1. As before, the functions $\mathfrak{F}_n$ and $\mathfrak{G}_n$ tend to $\mathfrak{a}_n$ and $K^2\mathfrak{a}_n$ as we move from the tail into the body of the lee-wave train. $\mathfrak{F}_n$ and $\mathfrak{G}_n$ are still defined by expressions of the form of (3.26) and (3.31 b), with an $\mathfrak{f}_n$

having the same qualitative properties as before, and with

$$c_n(k) \equiv K(k^2+j_n^2)^{-\frac{1}{2}}, \tag{A 7}$$

$$\gamma_n(k) \equiv Kj_n^2(k^2+j_n^2)^{-\frac{3}{2}}, \tag{A 8}$$

$$C_n \equiv c_n(0) = \gamma_n(0) = K/j_n; \quad C_n^{\pm} \equiv 1 \pm C_n, \tag{A 9}$$

$$\mathfrak{k}_n \equiv (K^2-j_n^2)^{\frac{1}{2}} \quad (n \leqslant n_K), \tag{A 10}$$

$$\mathfrak{v}_n \equiv 1-\gamma_n(\mathfrak{k}_n) \equiv \mathfrak{k}_n^2/K^2 > 0 \quad (n \leqslant n_K); \tag{A 11}$$

n_K is now defined by

$$j_{n_K} < K, \quad j_{n_K+1} > K. \tag{A 12}$$

In place of (3.1) we assume

$$j_1 < K \mp j_n \quad (n = 1, 2, \ldots). \tag{A 13}$$

By a calculation similar to that which led to (4.6a), we obtain

$$\epsilon^2 \mathfrak{e}_{nn}\left[(\tfrac{1}{2}\mathfrak{k}_n^2\,\partial_x|\mathfrak{F}_n^2|)\frac{d}{dr}[\{J_1(j_nr)\}^2]+O(t^{-1}\hat{\mathfrak{x}}_n)\right], \tag{A 14a}$$

for the contribution to the right-hand side of (A 1a) due to the nth tail, where $\mathfrak{e}_{nn}$ is still defined by (4.2). The non-Jacobian term on the right of (A 1a) does not contribute, to leading order. For the right-hand side of (A 1b) we obtain

$$\epsilon^2\mathfrak{e}_{nn}\left[(-\tfrac{1}{4}\mathfrak{k}_n^2\,\partial_x|\mathfrak{F}_n^2|)\frac{1}{r^2}\frac{d}{dr}[r^2\{J_1(j_nr)\}^2]+O(t^{-1}\hat{\mathfrak{x}}_n)\right], \tag{A 14b}$$

since (3.32) remains true.

Now let

$$\frac{d}{dr}[\{J_1(j_nr)\}^2] = \sum_{q=1}^{\infty}\alpha_{nq}J_1(j_qr), \tag{A 15a}$$

$$\frac{1}{r^2}\frac{d}{dr}[r^2\{J_1(j_nr)\}^2] = \sum_{q=1}^{\infty}\beta_{nq}J_1(j_qr). \tag{A 15b}$$

Of particular interest is

$$\alpha_{11} = -\tfrac{1}{2}\beta_{11} = -\frac{4}{3}\int_0^1\{J_1(j_1r)\}^3\,dr/\{J_0(j_1)\}^2$$
$$= -0{\cdot}675. \tag{A 16}$$

The fact that this is non-zero, in contrast to $\int \sin(2z)\sin z\,dz$, is what will allow upstream influence to occur in this problem even when $n_K = 1$.

Some further values of α_{nq} and β_{nq} are given in table 1. For each n, it can be shown that α_{nq} and β_{nq} are $O(q^{-2})$ as $q \to \infty$, implying uniform absolute convergence of (A 15).

We now have a problem of exactly the same form as (4.8), for each (n, q) $(n \leqslant n_K)$. The analysis of § 4.3 applies to each such problem. In place of (4.17) and (4.19) we have, for the qth pair of columnar disturbances $(1 \leqslant q \leqslant \infty)$ due to the nth lee-wave tail $(1 \leqslant n \leqslant n_K)$, that when $x \sim Vt$ $(V \neq \mathfrak{v}_n,\ V \neq C_q^{\mp})$:

q	$n=1$	2	3	4	5	6
1	−0·675 1·350	−0·547 0·642	−0·401 0·431	−0·313 0·326	−0·255 0·263	−0·216 0·220
2	2·274 3·363	−0·545 1·091	−0·536 0·704	−0·452 0·525	−0·382 0·420	−0·328 0·351
3	0·177 0·070	0·116 2·257	−0·601 1·201	−0·605 0·854	−0·542 0·670	−0·479 0·553
4	−0·057 −0·011	3·832 4·769	−0·238 1·673	−0·564 1·128	−0·584 0·868	−0·546 0·710
5	0·028 0·003	0·231 0·131	0·903 2·907	−0·380 1·613	−0·589 1·177	−0·609 0·941
6	−0·017 −0·001	−0·070 −0·024	5·018 5·842	0·177 2·115	−0·424 1·458	−0·570 1·140

TABLE 1. Some values of α_{nq} (top number of each entry) and β_{nq} (bottom number); n corresponds to the tail, and q to the modal structure of the columnar disturbance. These values are sufficient for calculating theoretical upstream velocity profiles when $n_K \leqslant 6$

$$\text{stream function} \sim \epsilon^2 (j_q)^{-1} \Gamma^{\pm}_{nq} r J_1(j_q r) \quad [V \in (\mathfrak{v}_n, C^{\pm}_q)], \tag{A 17a}$$

$$\text{scaled azimuthal velocity } (\chi) \sim \mp \epsilon^2 j_q C_q \Gamma^{\pm}_{nq} J_1(j_q r) \quad [V \in (\mathfrak{v}_n, C^{\pm}_q)], \tag{A 17b}$$

both being zero if $V \notin (\mathfrak{v}_n, C^{\pm}_q)$, where

$$\Gamma^{\pm}_{nq} \equiv \frac{\mathfrak{k}_n^2 |\mathfrak{a}_n^2| \{2\alpha_{nq} C_q \mp \beta_{nq}\}}{8 j_q C_q |C^{\pm}_q - \mathfrak{v}_n|}. \tag{A 17c}$$

In particular:

(i) The total upstream influence is obtained by choosing the lower signs and summing over both n and q, from 1 to n_K.

(ii) When $n_K = 1$ ($3{\cdot}83 < K < 7{\cdot}02$), the values of K giving an anomalously strong columnar disturbance just downstream of the lee waves are those for which C^-_2 is just greater than $\mathfrak{v}_1$, as in the two-dimensional case. From (A 9) and (A 11) it can be seen that this corresponds to

$$K \text{ just less than } j_1^{\frac{2}{3}} j_2^{\frac{1}{3}}, = 4{\cdot}69. \tag{A 18}$$

The columnar disturbances just downstream of the tail are then dominated by the contribution for which $n = 1, q = 2$ in (A 17) and the lower signs are chosen.

The foregoing are the main results analogous to those of §§4.5 and 4.6 (i). For the slowly-varying problem analogous to that of §4.6 (ii), the solution (4.20) applies, with precisely similar modifications, namely replacement of $\Gamma^{\pm}_{2n}$ by $\Gamma^{\pm}_{nq}$, of $2n$ by j_q, and of $\sin(2nz)$ by $rJ_1(j_q r)$ for ψ and by $J_1(j_q r)$ for χ. The summation in (4.20) becomes a double summation, from $n = 1$ to n_K and $q = 1$ to ∞.

Appendix B. Justification for the use of the long-wave solution

If $\Psi'_{2n} \equiv \Psi_{2n} - \overline{\Psi}_{2n}$ is the difference between the solution $\overline{\Psi}_{2n}$ of a (long-wave) problem of form (4.8), and a solution Ψ_{2n} of the corresponding full problem, then Ψ'_{2n} is defined by the difference problem

$$(\partial_t + \partial_x)(\partial_x^2 - 4n^2)\Psi'_{2n} + \partial_x \Theta'_{2n} = \partial_x^2 m, \tag{B 1a}$$

$$(\partial_t + \partial_x)\Theta'_{2n} - K^2 \partial_x \Psi'_{2n} = 0, \tag{B 1b}$$

$$\Psi'_{2n} = \Theta'_{2n} = 0 \quad \text{for} \quad t < 0, \tag{B 1c}$$

$$\partial_x \Psi'_{2n}, \quad \Theta'_{2n} \to 0 \quad \text{as} \quad |x| \to \infty \quad (t < \infty), \tag{B 1d}$$

where

$$m(x,t) \equiv -(\partial_t + \partial_x)\overline{\Psi}_{2n}. \tag{B 2}$$

The differentiated form of the forcing term, $\partial_x^2 m$, makes the absence of columnar disturbances from Ψ'_{2n} and Θ'_{2n} almost obvious, for forms of m likely to be of interest. A precise result is easily obtained as follows; it is sufficient for our purposes, although by no means the strongest possible. (It is worth emphasizing that the argument makes no reference at all to whether or not the long-wave solution is a good approximation to the full solution in all respects; the forcing terms do not have to be slowly varying. This fact is basic, for instance, to the reasoning summarized in §5.)

Let

$$\Psi_{\mathrm{I}} \equiv \int_{-\infty}^{x} d\Xi \int_{-\infty}^{\Xi} d\xi\, \Psi'_{2n}(\xi, z, t), \tag{B 3}$$

and similarly Θ_{I}. If $m \to 0$ sufficiently rapidly as $|x| \to \infty$, it is evident that Ψ_{I} and Θ_{I} comprise the solution of a problem of the form (B 1), but in which m replaces $\partial_x^2 m$. By multiplying the first equation of this latter problem by $-\Psi_{\mathrm{I}}$, the second by $K^{-2}\Theta_{\mathrm{I}}$, adding, integrating over all x, and using the boundary conditions, we obtain the corresponding energy relation

$$\partial_t(\tfrac{1}{2}\mathscr{E}^2) = -\int_{-\infty}^{\infty} \Psi_{\mathrm{I}}\, m\, dx,$$

where

$$\mathscr{E} \equiv \left[\int_{-\infty}^{\infty} \{4n^2\Psi_{\mathrm{I}}^2 + (\partial_x \Psi_{\mathrm{I}})^2 + K^{-2}\Theta_{\mathrm{I}}^2\}\, dx\right]^{\frac{1}{2}} \geqslant 0.$$

By Schwarz' inequality,

$$-\int_{-\infty}^{\infty} \Psi_{\mathrm{I}}\, m\, dx \leqslant \left[\int_{-\infty}^{\infty} \Psi_{\mathrm{I}}^2\, dx\right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} m^2\, dx\right]^{\frac{1}{2}}.$$

Also (crudely),

$$\left[\int_{-\infty}^{\infty} \Psi_{\mathrm{I}}^2\, dx\right]^{\frac{1}{2}} \leqslant \mathscr{E}.$$

Combining these relations, we get

$$\partial_t \mathscr{E} \leqslant \left[\int_{-\infty}^{\infty} m^2\, dx\right]^{\frac{1}{2}}, \tag{B 4}$$

whence, since $\mathscr{E} = 0$ for $t < 0$,

$$\mathscr{E} \leqslant \int_0^t \left[\int_{-\infty}^{\infty} m^2(x, \tau)\, dx\right]^{\frac{1}{2}} d\tau. \tag{B 5}$$

Now suppose that Ψ'_{2n} or Θ'_{2n} contains columnar disturbances whose strengths are of order ι. That is (§2.3), there is at least one region, whose length in the x-direction is increasing like t, throughout which Ψ'_{2n} or Θ'_{2n} is equal to a one-signed contribution of order ι plus, possibly, superposed oscillations of wavelength $o(t)$. From (B 3), Ψ_{I} or Θ_{I} must then be at least of order ιt^2 in the same region (its graph against x must be approximately a parabola). It follows that $\mathscr{E}$ must be at least of order $\iota t^{\frac{5}{2}}$. Hence

$$\iota = O\left[t^{-\frac{5}{2}}\int_0^t\left\{\int_{-\infty}^{\infty} \mathscr{m}^2(x,\tau)\,dx\right\}^{\frac{1}{2}}d\tau\right] \quad \text{as} \quad t\to\infty. \tag{B 6}$$

This provides an estimate governing the possible asymptotic order of magnitude of the strength of any columnar disturbance present in Ψ'_{2n} or Θ'_{2n}. It is presumably not sharp; the inequalities are crude, as is the use of the energy integral itself.

It is now easy, for instance, to justify the use of the long-wave formula in §4 to describe the columnar disturbances due to the self-interaction of a lee-wave tail, to within the approximations made there. In that case,

$$\mathscr{m} = -(\partial_t+\partial_x)\,\overline{\Psi}_{2n} = \tfrac{1}{4}n^{-2}\Lambda_{2n}\{(\partial_t+\partial_x)\mathfrak{M}_{nn}, (\partial_t+\partial_x)\,\mathfrak{B}_{nn}\}. \tag{B 7}$$

It can be shown that $\mathscr{m} = O(t^{\frac{1}{2}})$, crudely, under the assumption (4.2b), and that $\mathscr{m} = 0$ outside a domain of size $O(t)$ in the x-direction. Thence

$$\iota = O(t^{-\frac{1}{2}}),$$

which is smaller than other errors incurred in our main analysis.

The estimate $\mathscr{m} = O(t^{\frac{1}{2}})$ is based on the explicit results of §4, together with estimates of the contribution from the terms not explicitly shown in (4.6). The main principles that were used in deriving it are, first, that the operations ∂_t and ∂_x do not increase any orders of magnitude involved – this follows from use of (3.30), etc. – and, second, that the ranges of integration in (4.10b) are $O(t)$ in size.

REFERENCES

Abramowitz, M. & Stegun, I. A. (eds.) 1964 *Handbook of Mathematical Functions*. Nat. Bureau of Standards, Washington D.C.

Benjamin, T. B. 1970 Upstream influence. *J. Fluid Mech.* **40**, 49–79.

Benjamin, T. B. & Barnard, B. J. S. 1964 A study of the motion of a cavity in a rotating liquid. *J. Fluid Mech.* **19**, 193–209.

Bretherton, F. P. 1967 The time-dependent motion due to a cylinder moving in an unbounded rotating or stratified fluid. *J. Fluid Mech.* **28**, 545–570.

Bretherton, F. P. 1969 On the mean motion induced by internal gravity waves. *J. Fluid Mech.* **36**, 785–803.

Greenspan, H. P. 1968 *The Theory of Rotating Fluids*, pp. 26, 213–216. Cambridge University Press.

Jeffreys, H. & Jeffreys, B. S. 1962 *Methods of Mathematical Physics* (3rd edn.). Cambridge University Press.

Keady, G. 1971 Upstream influence in a two-fluid system. *J. Fluid Mech.* **49**, 373–384.

Lighthill, M. J. 1958 *Fourier Analysis and Generalized Functions*. Cambridge University Press.

Longuet-Higgins, M. S. & Stewart, R. W. 1964 Radiation stresses in water waves; a physical discussion, with applications. *Deep-Sea Res.* **11**, 529–562.

McEwan, A. D. 1971 Degeneration of resonantly-excited standing internal gravity waves. *J. Fluid Mech.* **50**, 431–448.

McIntyre, M. E. 1972 Mean motions and impulse of a horizontally-guided internal-gravity-wave packet. (Submitted to *J. Fluid Mech.*)

Martin, S., Simmons, W. F. & Wunsch, C. I. 1972. The excitation of resonant triads by single internal waves. *J. Fluid Mech.* (in the Press).

Miles, J. W. 1970 The oseenlet as a model for separated flow in a rotating viscous liquid. *J. Fluid Mech.* **42**, 207–217.

Miles, J. W. 1972 Upstream influence of a dipole in a rotating flow. (Submitted to *J. Fluid Mech.*)

Miles, J. W. & Huppert, H. E. 1969 Lee waves in a stratified flow. Part 4. Perturbation approximations. *J. Fluid Mech.* **35**, 497–525.

Morgan, G. W. 1951 A study of motions in a rotating liquid. *Proc. R. Soc.* A **206**, 108–130.

Pao, Y.-H. 1969 Inviscid flows of stably stratified fluids over barriers. *Quart J. Roy. Met. Soc.* **95**, 104–119.

Pedley, T. J. 1969 On the instability of viscous flow in a rapidly rotating pipe. *J. Fluid Mech.* **35**, 97–116.

Phillips, O. M. 1966 *The Dynamics of the Upper Ocean*, p. 41. Cambridge University Press.

Stewartson, K. 1952 On the slow motion of a sphere along the axis of a rotating fluid. *Proc. Camb. Phil. Soc.* **48**, 168–177.

Stewartson, K. 1970 A note on forward wakes in rotating fluids. *J. Fluid Mech.* **42**, 219–223.

Trustrum, K. 1964 Rotating and stratified fluid flow. *J. Fluid Mech.* **19**, 415–432.

Trustrum, K. 1971 An Oseen model of the two-dimensional flow of a stratified fluid over an obstacle. *J. Fluid Mech.* **50**, 177–188.

Wong, K. K. & Kao, T. W. 1970 Stratified flow over extended obstacles and its application to topographical effect on vertical wind shear. *J. Atmos. Sci.* **27**, 884–889.

Yih, C.-S. 1965 *Dynamics of Inhomogeneous Fluids*, pp. 77, 82, 94. Macmillan.